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RECOGNIZING TRACE GRAPHS OF CLOSED BRAIDS

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Abstract

To a closed braid in a solid torus we associate a trace graph in a thickened torus in such a way that closed braids are isotopic if and only if their trace graphs can be related by trihedral and tetrahedral moves. For closed braids with a fixed number of strands, we recognize trace graphs up to isotopy and trihedral moves in polynomial time with respect to the braid length.

1. Introduction

1.1. Motivation and summary. There is still no efficient solution to the conjugacy problem for braid groups B_n on $n \geq 5$ strands, i.e. with a polynomial complexity in the braid length. Very promising steps towards a polynomial solution were made by Birman, Gebhardt, González-Meneses [2, 3, 4] and Ko, Lee [13]. A clear obstruction is that the number of different conjugacy classes of braids grows exponentially even in B_3, see Murasugi [14].

The conjugacy problem for braids is equivalent to the isotopy classification of closed braids in a solid torus. To a closed braid in a solid torus we associate a 1-parameter family of closed braids, which is encoded by the labelled *trace graph* in a thickened torus. We call this construction a 1-*parameter* approach to links.

We establish the higher order Reidemeister theorem for closed braids: *trace graphs determine families of isotopic closed braids if and only if they can be related by a finite sequence of the trihedral and tetrahedral moves shown in* Fig. 5 *and* Fig. 6, see Theorem 1.4. We recognize trace graphs of closed braids up to isotopy in a thickened torus and trihedral moves in polynomial time with respect to the braid length, see Theorem 1.5. This is one of very few known polynomial algorithms recognizing complicated topological objects up to isotopy.

1.2. Basic definitions of braid theory. We work in the C^∞-smooth category. To explain important constructions we may draw piecewise linear pictures that can be easily smoothed. Fix Euclidean coordinates x, y, z in $\mathbb{R}^3$. Denote by D_{xy} the unit disk at the origin 0 of the horizontal plane XY. Introduce the *solid torus* $V = \mathrm{D}_{xy} \times S^1_z$, where the oriented circle S^1_z is the segment $[-1, 1]_z$ with the identified endpoints, see the left picture of Fig. 1.

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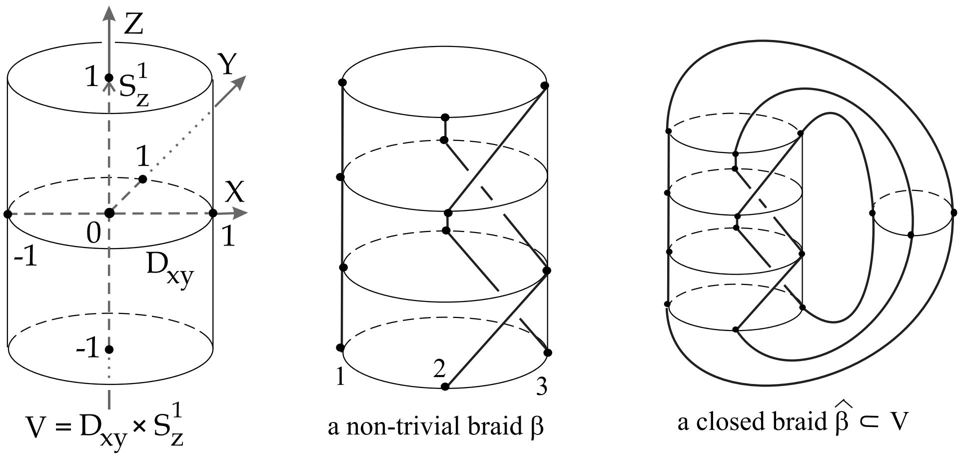


Fig. 1. A braid and its closure in the solid torus V.

DEFINITION 1.1. Mark n points $p_1, \ldots, p_n \in \mathrm{D}_{xy}$. A *braid* β on n *strands* is the image of a smooth embedding of n segments into $\mathrm{D}_{xy} \times [-1,1]_z$ such that

- the strands of β are monotonic with respect to $\mathrm{pr}_z \colon \beta \to S^1_z$ (see Fig. 1);
- the lower and upper endpoints of β are $\bigcup(p_i \times \{-1\})$, $\bigcup(p_i \times \{1\})$, respectively.

Identifying the bases $\mathrm{D}_{xy} \times \{z = \pm 1\}$, the cylinder $\mathrm{D}_{xy} \times [-1,1]_z$ is converted into the solid torus $V = \mathrm{D}_{xy} \times S^1_z$, while a braid $\beta \subset \mathrm{D}_{xy} \times [-1,1]_z$ becomes the *closed* braid $\hat{\beta} \subset V$, see the right picture of Fig. 1.

DEFINITION 1.2. Braids are considered up to an *isotopy*, a smooth deformation of the cylinder $\mathrm{D}_{xy} \times [-1,1]_z$, fixed on its boundary. The equivalence classes of braids form the group denoted by B_n. The *product* of braids β_1, β_2 is the braid $\beta_1\beta_2$ obtained by attaching a cylinder containing β_2 over a cylinder containing β_1. The *trivial* braid consists of n vertical straight segments $\bigsqcup_{i=1}^{n}(p_i \times [-1,1]_z)$.

The braid group B_n is generated by elementary braids σ_i, $i = 1, \ldots, n-1$, where σ_i is a right half-twist of strands $i, i+1$, the remaining strands are vertical. The braid β in the middle picture of Fig. 1 is isotopic to σ_2^2. Any braid induces a *permutation* of its endpoints, e.g. the braid β induces the trivial permutation on 1, 2, 3. Such a braid $\beta \in B_n$ is called *pure* and its closure consists of n components.

1.3. Trace graphs of closed braids. Closed braids are usually represented by plane diagrams with double crossings. A classical approach to the isotopy classification of closed braids is to use diagram invariants, i.e. functions defined on plane diagrams and invariant under Reidemester moves II, III in Fig. 2. A 1-parameter approach proposed by Fiedler and Kurlin [9] is to consider the 1-parameter family of diagrams of

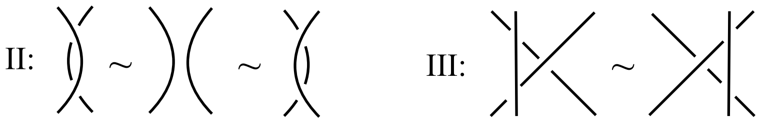


Fig. 2. Reidemeister moves on braids.

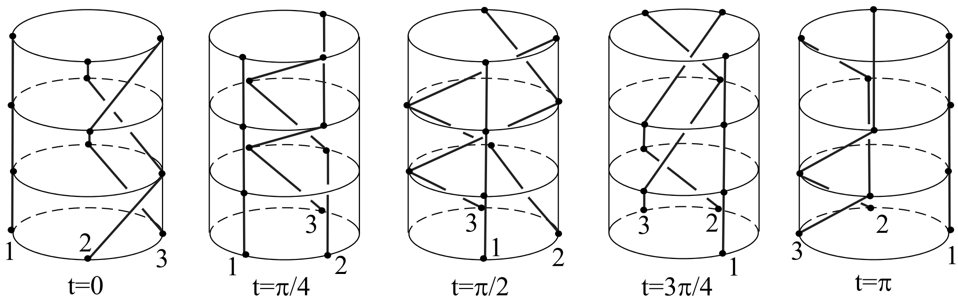


Fig. 3. Diagrams of rotated braids $\mathrm{rot}_t(\beta)$ for the braid β in Fig. 1.

braids rotated around the core of the solid torus V. This family contains more combinatorial information about a closed braid than just one plane diagram and involves such features of braids as meridional *trisecants*, straight lines meeting a braid in 3 points and contained in a meridional disk $\mathrm{D}_{xy} \times \{z\}$ of the solid torus V.

A *long knot* in $\mathbb{R}^3$, a single curve approaching the vertical axis Z at $\pm\infty$, can be also rotated in $\mathbb{R}^3$ around Z, but closed braids are more naturally rotated in V. It is essential to work in the solid torus instead of $\mathbb{R}^3$ since our 1-parameter family represents a non-trivial rational homology class in the space of all diagrams. A. Hatcher has proven that the space of diagrams of a prime knot in $\mathbb{R}^3$ has a finite fundamental group [12]. Consequently, its rational first homology group vanishes.

DEFINITION 1.3. Given a closed braid $\hat{\beta} \subset V$ in a general position (see more details in Subsection 2.1), consider *rotated* braids $\mathrm{rot}_t(\hat{\beta}) \subset V$ obtained by the rotation of $\hat{\beta}$ through an angle $t \in [0, 2\pi)$. Project each of the rotated braids $\mathrm{rot}_t(\hat{\beta})$ to the fixed annulus $A_{xz} = [-1, 1]_x \times S^1_z \subset V$, see Fig. 3. The crossings of the resulting diagrams form the *trace graph* $\mathrm{TG}(\hat{\beta})$ that lives in the *thickened* torus $\mathbb{T} = A_{xz} \times S^1_t$, see Fig. 4, where the time circle S^1_t is $[0, 2\pi]$ with the identified endpoints.

For the braid β from Fig. 1, a triple point occurs in a diagram of $\mathrm{rot}_t(\beta)$, $t \in (\pi/4, \pi/2)$, where strand 1 crosses over strand 3, which crosses over strand 2. The associated vertex of $\mathrm{TG}(\hat{\beta})$ and its image under $t \mapsto t + \pi$ are in Fig. 4.

Label arcs of a pure braid $\beta \in B_n$ by $1, 2, \ldots, n$ as in the middle picture of Fig. 1. Any general point p of the trace graph $\mathrm{TG}(\hat{\beta}) \subset \mathbb{T}$ is a crossing of arcs i, j in the

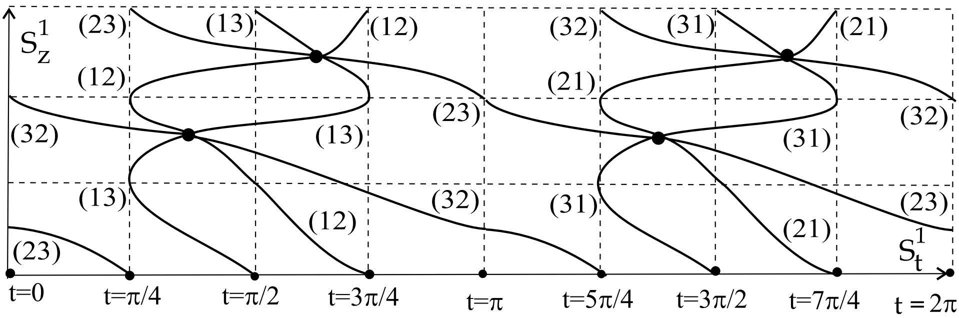


Fig. 4. The trace graph of the closed braid $\hat{\beta}$ in Fig. 1.

diagram of a rotated braid $\mathrm{rot}_t(\hat{\beta})$, i.e. the point p evolves in $\mathbb{T}$ following a trace of crossings in the diagrams. Label the point p by the ordered pair (ij) if the arc i is over the arc j in the diagram of $\mathrm{rot}_t(\hat{\beta})$ and by the ordered pair (ji) otherwise. For non-pure braids, other well-defined markings will be introduced in Definition 3.2. The trace graph maps to itself under the time shift $t \mapsto t + \pi$, each label (ij) reverses to (ji). Each labelled closed loop of $\mathrm{TG}(\hat{\beta})$ is monotonic with respect to the vertical circle S^1_z, but not with respect to the time circle S^1_t.

The trace graph of the piecewise linear closed braid $\hat{\beta}$ in Fig. 1 is projected to the torus $\mathrm{ZT} = S^1_z \times S^1_t$ and is shown in Fig. 4. A vertical section $\mathrm{TG}(\hat{\beta}) \cap (\mathrm{A}_{xz} \times \{t\})$ of a trace graph consists of finitely many points, which are crossings of the diagram of $\mathrm{rot}_t(\hat{\beta})$, e.g. the zero section $\mathrm{TG}(\hat{\beta}) \cap (\mathrm{A}_{xz} \times \{0\})$ contains 2 points associated to the crossings of the original braid $\hat{\beta}$. The section $\mathrm{TG}(\hat{\beta}) \cap (\mathrm{A}_{xz} \times \{\pi/4\})$ has 2 *tangent* vertices, when the rotated braid $\mathrm{rot}_{\pi/4}(\hat{\beta})$ has 2 simple tangencies (arc 1 over arcs 2, 3), so the diagram of $\mathrm{rot}_{\pi/4}(\hat{\beta})$ changes under Reidemeister moves II.

The braid β has two meridional trisecants associated to two *triple* vertices of $\mathrm{TG}(\hat{\beta})$. Under the rotation of β through some $t \in (\pi/4, \pi/2)$ and $t \in (\pi/2, 3\pi/4)$, the trisecants become perpendicular to the plane of projection, so triple intersections appear in the corresponding diagrams of $\mathrm{rot}_t(\hat{\beta})$. Around these singular moments the diagrams change under Reidemeister moves III, notice that the labels don't change at triple points, see more details about singularities and general position of braids in Subsection 2.1. A given closed braid can be reconstructed from its trace graph with labels, see also combinatorial constructions of a trace graph in Subsection 2.2.

Theorem 1.4. *Closed braids $\hat{\beta}_0, \hat{\beta}_1$ are isotopic in the solid torus V if and only if their labelled trace graphs* $\mathrm{TG}(\hat{\beta}_0), \mathrm{TG}(\hat{\beta}_1) \subset \mathbb{T}$ *can be obtained from each other by an isotopy in $\mathbb{T}$ and a finite sequence of moves in* Fig. 5 *and* Fig. 6.

The trihedral move is associated to the singular situation in the space of all closed braids, when the path of rotated braids touches the singular subspace of triple inter-

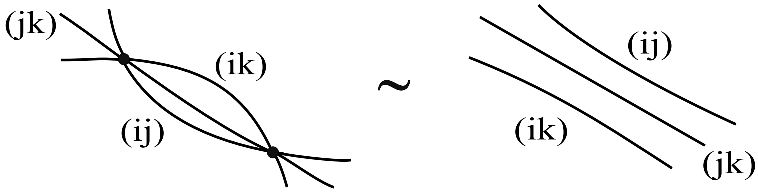


Fig. 5. Trihedral move on trace graphs.

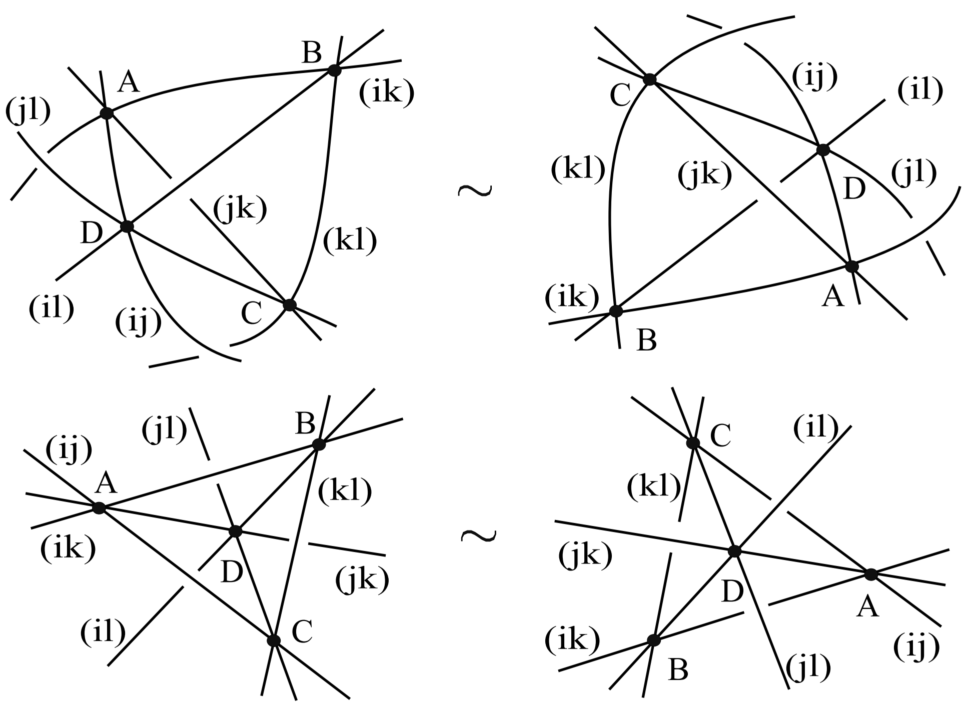


Fig. 6. Tetrahedral moves on trace graphs.

sections, i.e. under the rotation 3 crossings approach each other as in Reidemeister move III, but then go back in the reverse direction without completing Reidemeister move III. The tetrahedral move is associated to passing through the singular subspace of quadruple intersections, when a 1-skeleton of some tetrahedron collapses in the trace graph to a point and then blows up again in a symmetric form.

Theorem 1.4 can be used to construct invariants of closed braids reflecting such geometric features as meridional trisecants. Similar easily computable lower bounds on the number of fiber quadrisecants in knot isotopies were found by Fiedler and Kurlin [8]. On the other hand trace graphs turned out to be complicated topological objects that can be recognized up to isotopy in a polynomial time.

Theorem 1.5. *Let $\beta, \beta' \in B_n$ be braids of length $\leq l$. There is an algorithm of complexity $C(n/2)^{n^2/8}(6l)^{n^2-n+1}$ to decide whether* $\mathrm{TG}(\hat{\beta})$ *and* $\mathrm{TG}(\hat{\beta}')$ *are related by isotopy in $\mathbb{T}$ and trihedral moves, the constant C does not depend on l and n. In the case of pure braids, the power $n^2/8$ can be replaced by 1. If the closure of a braid is a knot, a single circle in the solid torus, then the complexity reduces to $Cn(6l)^{n-1}$.*

2. Studying closed braids in terms of their trace graphs

2.1. Singularities and general position of closed braids. Here we outline of the proof of Theorem 1.4, which follows from a more general result by Fiedler and Kurlin [9, Theorem 1.4] on links in the solid torus V.

Codimension 1 singularities of closed braids with respect to the plane projection are tangencies of order 1 and triple intersections associated to Reidemeister moves II and III, respectively, see Fig. 2. The Reidemeister theorem says that any isotopy in the space SB of all closed braids (with respect to the Whitney topology) can be approximated by a path transversal to the singular subspace $\Sigma \cup \Sigma \subset \mathrm{SB}$. We extend this approach to 1-parameter families of rotated closed braids.

Codimension 2 singularities of plane diagrams of closed braids are quadruple points, tangent triple points and tangencies of order 2. A closed braid $\hat{\beta} \subset V$ can be put in a *general position* such that the *canonical* loop of rotated braids $\{\mathrm{rot}_t(\hat{\beta})\} \subset \mathrm{SB}$ is transversal to the codimension 1 subspace $\Sigma \cup \Sigma \subset \mathrm{SB}$ and avoids the codimension 2 subspace $\Sigma \cup \Sigma \cup \Sigma \subset \mathrm{SB}$. Similarly any isotopy of closed braids can be approximated by a path $\{\hat{\beta}_s\}_{s=0}^{s=1}$ such that the cylinder of canonical loops $\{\mathrm{rot}_t(\hat{\beta}_s)\}$ is transversal to $\Sigma \cup \Sigma \cup \Sigma \subset \mathrm{SB}$. Passing through these singularities leads to tetrahedral moves in Fig. 6, trihedral move in Fig. 5 and a move where a triple vertex of a trace graph passes through a tangent vertex, which does not change the combinatorial structure of the trace graph with labels, see more details in Fiedler and Kurlin [9].

A geometric interpretation of a trihedral move and tetrahedral move at the level of closed braids is shown in Fig. 7. In a tetrahedral move two arcs intersect a wide band bounded by other two arcs, so two intersection points swap their heights. The

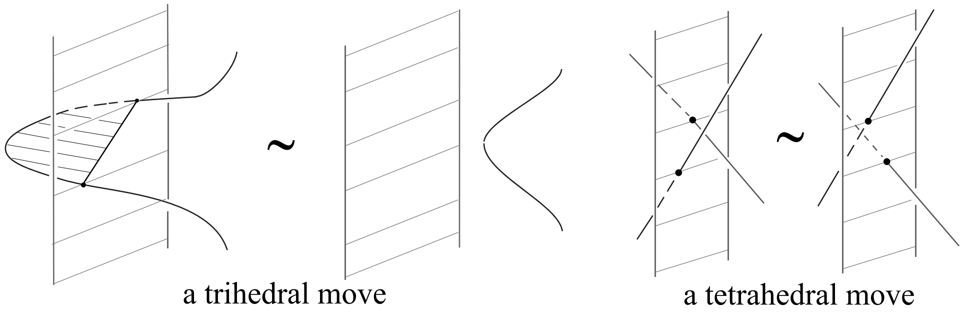


Fig. 7. A trihedral move and a tetrahedral move for braids.

first move in Fig. 6 applies when the intermediate oriented arcs go together from one side of the band to another like $\rightrightarrows$. The second move in Fig. 6 means that the arcs are antiparallel as in the British rail mark $\rightleftarrows$.

2.2. Combinatorial constructions of a trace graph. First we show how to create the trace graph using an algebraic form of a braid.

Lemma 2.1. *Let $\beta \in B_n$ be a braid of length l. Then the closure $\hat{\beta}$ is isotopic in the solid torus V to a closed braid whose trace graph contains $2l(n-2)$ triple vertices.*

Proof. Let $\Delta \in B_n$ be Garside's element [10], i.e. Δ^2 is a generator of the center of B_n, the full twist of n strands. The rotation of a braid $\beta \in B_n$ can be considered as a commutation of β with Δ^2. So the canonical loop of rotated closed braids $\mathrm{rot}_t(\hat{\beta})$ is represented by the sequence of the closures of the following braids:

$$\beta \to \Delta\Delta^{-1}\beta \to \Delta^{-1}\beta\Delta \to \Delta^{-1}\Delta\beta' \to \beta' \to \Delta\Delta^{-1}\beta' \to \Delta^{-1}\beta'\Delta \to \Delta^{-1}\Delta\beta \to \beta.$$

The first arrow in the sequence consists of Reidemester moves II creating couples of symmetric crossings. The second arrow represents an isotopy of the diagram when we push Δ through the trivial part of the closed braid $\hat{\beta}$, i.e. we cyclically shift the letters of $\Delta\Delta^{-1}\beta$ to get $\Delta^{-1}\beta\Delta$. The third arrow shows how Δ acts on β from the right. After we get a new braid β', we apply the same transformation and finish with β since $\beta\Delta = \Delta\beta'$ implies that $\beta'\Delta = \Delta\beta$.

For $n = 3$, we have $\Delta = \sigma_1\sigma_2\sigma_1$. We need to consider only the two generators σ_1, σ_2 and their inverses. We apply braid relations corresponding to Reidemeister moves II and III associated to tangent and triple vertices of $\mathrm{TG}(\beta)$.

$$\sigma_1\Delta = \sigma_1(\sigma_1\sigma_2\sigma_1) \to \sigma_1(\sigma_2\sigma_1\sigma_2) = \Delta\sigma_2,$$
$$\sigma_2\Delta = \sigma_2(\sigma_1\sigma_2\sigma_1) \to (\sigma_1\sigma_2\sigma_1)\sigma_1 = \Delta\sigma_1,$$
$$\sigma_1^{-1}\Delta = \sigma_1^{-1}(\sigma_1\sigma_2\sigma_1) \to \sigma_2\sigma_1 \to (\sigma_1\sigma_1^{-1})\sigma_2\sigma_1 \to \sigma_1(\sigma_2\sigma_1\sigma_2^{-1}) = \Delta\sigma_2^{-1},$$
$$\sigma_2^{-1}\Delta = \sigma_2^{-1}(\sigma_1\sigma_2\sigma_1) \to (\sigma_1\sigma_2\sigma_1^{-1})\sigma_1 \to \sigma_1\sigma_2 \to \sigma_1\sigma_2(\sigma_1\sigma_1^{-1}) = \Delta\sigma_1^{-1}.$$

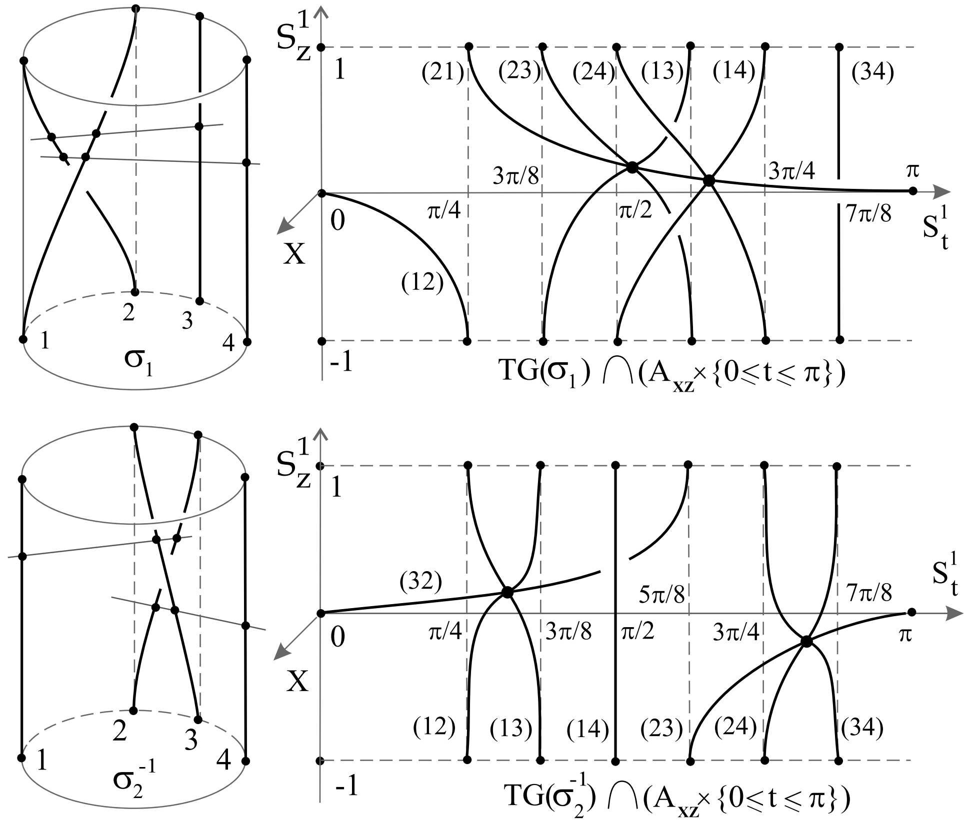


Fig. 8. Half trace graphs of the 4-braids $\sigma_1, \sigma_2^{-1} \in B_4$.

Notice that the sequence is canonical in the case of a generator and almost canonical in the case of an inverse generator. Indeed we can replace the above sequence by $\sigma_1^{-1}\Delta \to \Delta\sigma_2^{-1}$ by $\sigma_1^{-1}(\sigma_1\sigma_2\sigma_1) \to \sigma_2\sigma_1 \to \sigma_2\sigma_1(\sigma_2\sigma_2^{-1}) \to (\sigma_1\sigma_2\sigma_1)\sigma_2^{-1}$. Pushing Δ through a generator or its inverse creates exactly $n-2$ triple points. So we end up with $2l(n-2)$ triple points, because we push Δ twice through the braid. □

Now we construct a trace graph of a closed braid in a geometric way.

Lemma 2.2. *Let $\beta \in B_n$ be a braid of length l. Then the closure $\hat{\beta}$ is isotopic in the solid torus V to a closed braid whose trace graph consists of elementary blocks associated to the generators (and their inverses) of B_n similar to* Fig. 8.

Proof. Fig. 8 shows the trace graphs of the elements σ_1 and σ_2^{-1} in the braid group B_4. In general we mark out the points $\psi_k = 2^{1-k}\pi$, $k = 0, \dots, n-1$ on the boundary of the bases $D_{xy} \times \{\pm 1\}$. The 0-th point $\psi_0 = 2\pi$ is the n-th point.

The crucial feature of the distribution $\{\psi_k\}$ is that all straight lines passing through two points ψ_j, ψ_k are not parallel to each other. Firstly we draw all strands in the cylinder $\partial D_{xy} \times [-1,1]_z$. Secondly we approximate with the first derivative the strands forming a crossing by smooth arcs, see the left hand side pictures of Fig. 8.

Then each elementary braid σ_i constructed as above has exactly $n-2$ meridional trisecants, one trisecant through the strands $i, i+1$ and j for each $j \neq i, i+1$. Each trisecant is associated to a triple vertex of the trace graph, see 4 meridional trisecants in the left hand side pictures of Fig. 8. The right hand side pictures in Fig. 8 contain the trace graphs of the corresponding 4-braids. The braids are not in general position, e.g. parallel strands 3 and 4 lead to the vertical arc labelled with (34), but we may slightly deform such a braid, which makes the projection $\mathrm{TG} \to S^1_t$ generic. □

3. Two splittings of trace graphs of closed braids

3.1. A trace graph splits into trace circles.

Lemma 3.1. *For a braid $\beta \in B_n$, let $(n_1, \dots, n_m)$ be the lengths of cycles in the induced permutation $\tilde{\beta} \in S_n$. Number all components of the closure $\hat{\beta}$ by $1, \dots, m$. Set $N(\beta) = \sum_{i=1}^m (n_i - 1) + 2\sum_{i<j} \gcd(n_i, n_j)$, gcd is the greatest common divisor.*
(a) *The trace graph $\mathrm{TG}(\hat{\beta}) \subset \mathbb{T}$ splits into $N(\beta)$ circles such that each circle is a trace of crossings formed by 2 points simultaneously travelling along $\hat{\beta}$.*
(b) *If the braid β is pure, i.e. the permutation $\tilde{\beta}$ is trivial, then $N(\beta) = n(n-1)$. If the closure $\hat{\beta}$ of the braid β is a knot, then $N(\beta) = n - 1$.*
For any braid $\beta \in B_n$, we have $n - 1 \leq N(\beta) \leq n(n-1)$.

Proof. (a) Denote by $p_1, \dots, p_n$ the intersections of $\hat{\beta}$ with $D_{xy} \times \{-1\}$, ordered by the orientation of $\hat{\beta}$. Suppose that p_r, p_s belong to the q-th component of $\hat{\beta}$. This component corresponds to a cycle of the length n_q of the permutation $\tilde{\beta} \in S_n$.

If we push p_r, p_s along their strands in $\hat{\beta}$, the associated point in $\mathrm{TG}(\hat{\beta})$ goes along a circle and comes to the point corresponding to the next pair (say) (p_{r+1}, p_{s+1}). This process continues until we come to the original pair (p_r, p_s) after n_q steps along the cycle of $\tilde{\beta}$ having passed through n_q of $n_q(n_q - 1)$ ordered pairs. For each cycle of length n_q of $\tilde{\beta} \in S_n$, we get $n_q - 1$ circles that can be distinguished by non-zero differences $r - s \pmod{n_q} \in \{1, \dots, n_q - 1\}$.

Assume that p_r, p_s are in different components $i \neq j$ of $\hat{\beta}$, associated to cycles of lengths n_i, n_j. Then the process above terminates after $\mathrm{lcm}(n_i, n_j)$ steps, lcm is the lowest common multiple, since at each step indices r, s shift by 1 in two sets of lengths n_i, n_j. For any two cycles of lengths n_i, n_j in $\tilde{\beta}$, we get $2\gcd(n_i, n_j)$ circles split into pairs symmetric with respect to the time shift $t \mapsto t + \pi$.

(b) If $\hat{\beta}$ is a knot then the permutation $\tilde{\beta}$ is cyclic, i.e. $m = 1$, $N(\beta) = n - 1$. For a pure braid $\beta \in B_n$, we have $n_1 = \cdots = n_m = 1$, hence $N(\beta) = n(n-1)$. The upper

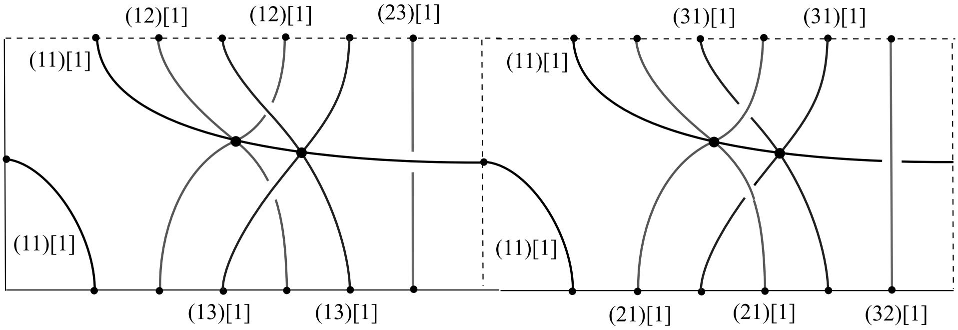


Fig. 9. Trace circles in $\mathrm{TG}(\hat{\sigma}_1)$ for the non-pure 4-braid $\sigma_1 \in B_4$.

estimate $N(\beta) \le n(n-1)$ geometrically follows from the fact that all circles from (a) are monotonic in the direction S^1_z and each meridional disk $\mathrm{D}_{xy} \times \{z\}$ intersects $\hat{\beta}$ in exactly n points leading to $n(n-1)$ crossings appearing under the rotation.

Let n_1 be minimal among all lengths $n_i > 1$. Under the map $(n_1, n_2, \dots, n_m) \mapsto (\overbrace{1, \dots, 1}^{n_1}, n_2, \dots, n_m)$, the number $N(\beta)$ of circles from (a) increases by

$$n_1(n_1-1) + 2n_1(m-1) - (n_1-1) - 2\sum_{i=2}^{m} \gcd(n_1, n_i)$$

$$\ge (n_1-1)^2 + 2n_1(m-1) - 2\sum_{i=2}^{m} n_1 = (n_1-1)^2 \ge 0.$$

So $N(\beta)$ is minimal if $\hat{\beta}$ is a knot and maximal if β is pure. □

DEFINITION 3.2. For a braid $\beta \in B_n$, a *trace* circle of the trace graph $\mathrm{TG}(\hat{\beta})$ is a circle consisting of crossings formed by 2 points simultaneously travelling along $\hat{\beta}$, e.g. a trace circle does not change its direction at triple vertices, compare Figs. 8 and 9. By Lemma 3.1 trace circles can be denoted by $T_{(ij)[k]}$ as follows. In the case $i = j$ the trace circles $T_{(ii)[k]}$ are associated to crossings formed by the points of $(\mathrm{D}_{xy} \times \{z\}) \cap \hat{\beta}$ from the i-th component of $\hat{\beta}$. If we index these points by $1, \dots, n_i$ according to the orientation of the i-th component then the number $k \in \{1, \dots, n_i - 1\}$ in the notation $(ii)[k]$ is well-defined as the difference between the indices modulo n_i. The trace circles $T_{(ij)[k]}$ with $i \ne j$ are generated by the i-th and j-th components of $\hat{\beta}$. Then $k \in \{1, \dots, \gcd(n_i, n_j)\}$ is defined up to cyclic permutation, i.e. if a trace circle is marked by 1, this defines markings on other circles $T_{(ij)[k]}, T_{(ji)[k]}$, $k > 1$.

EXAMPLE 3.3. Fig. 8 contains halfs of the trace graphs $\mathrm{TG}(\sigma_1), \mathrm{TG}(\sigma_2^{-1})$ of the non-pure braids $\sigma_1, \sigma_2^{-1} \in B_4$. The labels on edges in Fig. 8 correspond to numbers 1, 2,

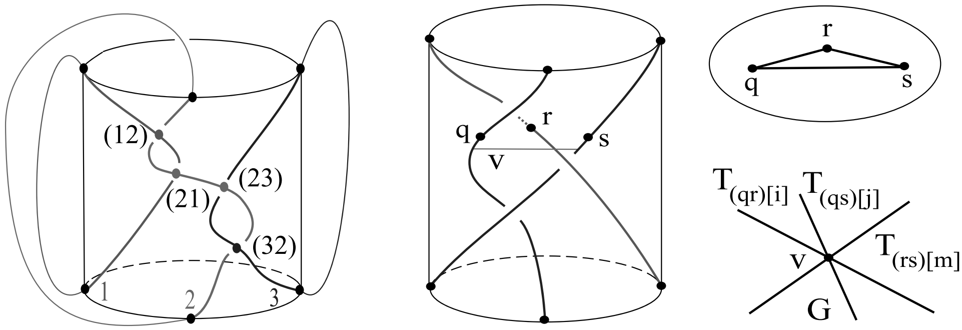


Fig. 10. Markings of crossings and trace circles.

3, 4 of the strands in the 4-braids. Fig. 9 shows the complete trace graph $\mathrm{TG}(\hat{\sigma}_1)$ split into trace circles with markings $(ij)[k]$ from Definition 3.2. The closure of $\sigma_1 \in B_4$ has $m = 3$ components, the lengths of cycles in the permutation $\tilde{\sigma}_1 \in S_4$ are $(n_1, n_2, n_3) = (2, 1, 1)$, hence $\mathrm{TG}(\hat{\sigma}_1)$ splits into $N(\sigma_1) = 7$ trace circles by Lemma 3.1 (a). One of these 7 trace circles is marked by (11)[1] and is formed by self-crossings of the first component consisting of strands 1, 2. The remaining 6 trace circles are marked by $(ij)[1]$ and are formed by crossings of different components, i.e. $i, j \in \{1, 2, 3\}$, $i \neq j$. Trace graphs of some pure braids are in Fig. 11.

If $\hat{\beta}$ is pure then the markings reduce to ordered pairs (ij), see Fig. 10. If the closure $\hat{\beta}$ is a knot, the markings $[k]$ have well-defined numbers $k \in \{1, \dots, n-1\}$.

Lemma 3.4. *Let $\beta \in B_n$ be a braid. Consider trace circles $T_{(qr)[i]}, T_{(qs)[j]}, T_{(rs)[m]}$ passing through a triple vertex $v \in \mathrm{TG}(\hat{\beta})$.*
(a) *The trace circle $T_{(qs)[j]}$ passes between $T_{(qr)[i]}$ and $T_{(rs)[m]}$ at the vertex v.*
(b) *If β is pure then each trace circle $T_{(ij)}$ maps to $T_{(ji)}$ under $t \mapsto t + \pi$. If $\hat{\beta}$ is a knot then each trace circle $T_{[m]}$ maps to $T_{[n-m]}$ under $t \mapsto t + \pi$.*

Proof. (a) Let the components of $\hat{\beta}$ indexed by q, r, s form a triple intersection (qrs) associated to the vertex $v \in \mathrm{TG}(\hat{\beta})$, see Fig. 10. Consider a disk $\mathrm{D}_{xy} \times \{z\}$ slightly above the triple intersection (qrs). In the disk we see 3 points of the arcs q, r, s. These points form a triangle, the angle at the point of r is close to π.

Denote by $t_{qr}, t_{rs}, t_{qs} \in S^1_t$ the time moments when the corresponding arcs in the closed braid $\hat{\beta}$ form a crossing under pr_{xz}. Then t_{qs} is between t_{qr} and t_{rs}. So the crossing (qs) is associated to the middle circle $T_{(qs)[j]}$ between $T_{(qr)[i]}$ and $T_{(rs)[m]}$.

(b) For a pure braid β, let a point $p \in T_{(ij)}$ correspond to a crossing $(p_i, p_j) \subset \hat{\beta} \cap (\mathrm{D}_{xy} \times \{z\})$. Under $t \mapsto t + \pi$, the crossing (p_i, p_j) converts to the reversed crossing (p_j, p_i) associated to the trace circle marked by (ji), see Fig. 11.

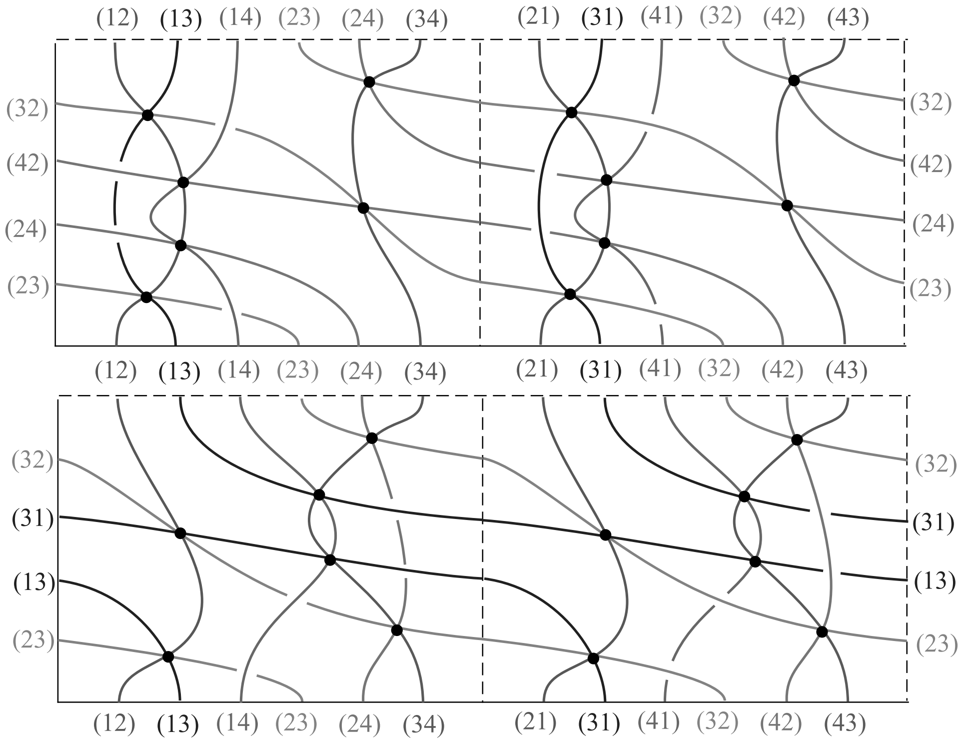


Fig. 11. Splittings of $\mathrm{TG}(\widehat{\sigma_2\sigma_3^2\sigma_2})$, $\mathrm{TG}(\widehat{\sigma_2\sigma_1^2\sigma_2})$ into trace circles.

If $\hat{\beta}$ is a knot, order all intersections $(p_1, \dots, p_n) \subset \hat{\beta} \cap (\mathrm{D}_{xy} \times \{z\})$ according to the orientation of $\hat{\beta}$. Then the marking $[m]$ of a crossing (p_r, p_s) is $r - s \pmod{n}$, hence the reversed crossing (p_s, p_r) has the marking $s - r \equiv n - m \pmod{n}$. □

3.2. A trace graph splits into level subgraphs. Here we split the trace graph $\mathrm{TG}(\hat{\beta})$ of a closed braid $\hat{\beta}$ into level subgraphs, trivalent graphs $S^{(k)}$, where $\beta \in B_n$ and $k = 1, \dots, n-1$.

DEFINITION 3.5. Any point $p \in \mathrm{TG}(\hat{\beta})$ that is not a vertex corresponds to an ordered pair $(p_i, p_j) \subset \hat{\beta} \cap (\mathrm{D}_{xy} \times \{z\})$. Let t be the time moment when p_i, p_j project to the same point under $\mathrm{pr}_{xz}\colon \hat{\beta} \to \mathrm{A}_{xz}$. Then $\mathrm{rot}_t(\hat{\beta})$ in a thin slice $\mathrm{D}_{xy} \times (z-\varepsilon, z+\varepsilon)$ looks like a braid generator σ_k or σ_k^{-1}, all other strands do not cross each other, see Fig. 12. The index k is called *the level* of the point $p \in \mathrm{TG}(\hat{\beta})$.

From another point of view we may compute the level of a crossing (p_i, p_j) as follows. Take the oriented straight segment d having endpoints on $\partial \mathrm{D}_{xy} \times \{z\}$ and passing through p_i first and p_j after. Complete d with the arc of $\partial \mathrm{D}_{xy} \times \{z\}$ to get an oriented

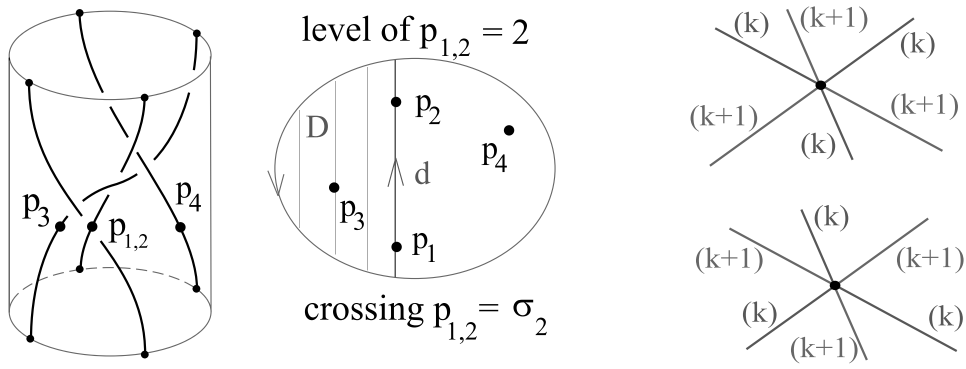


Fig. 12. Levels in trace graphs of closed braids.

circuit bounding a disk D, see Fig. 12. The number of intersections $\hat{\beta} \cap \operatorname{Int} D$ plus 1 is called the *level* (k) of the point p. We chose the name *level* since any crossing of $\mathrm{pr}_{xz}(\hat{\beta})$ is located at its horizontal level with respect to X.

Lemma 3.6. *Going along a trace circle of a trace graph, the level of a point* p *may change only at a triple vertex as follows*: $k \mapsto k \pm 1$, *see* Fig. 12.

Proof. The number of intersections of $\hat{\beta}$ with the disk D from Definition 3.5 remains invariant until the segment d passes through other points of $\hat{\beta} \cap (\mathrm{D}_{xy} \times \{z\})$ apart from p_i, p_j defining $p \in \mathrm{TG}(\hat{\beta})$. While p passes through a triple vertex of $\mathrm{TG}(\hat{\beta})$, the segment d intersects exactly one strand of β, hence the number of points $\hat{\beta} \cap D$ changes by ± 1. This also follows from $\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}$. □

Orient the 2-dimensional torus $\mathrm{ZT} = S^1_z \times S^1_t$ in such a way that the first direction is *vertical* along S^1_z and the second one is *horizontal* opposite to S^1_t.

DEFINITION 3.7. Let $\mathrm{TG}(\hat{\beta})$ be the trace graph of a closed braid $\hat{\beta}$, where $\beta \in B_n$. For each $k = 1, \ldots, n-1$, denote by $S^{(k)}$ the k-th *level subgraph* consisting of all edges having the level k. Orient each edge of $\mathrm{TG}(\hat{\beta})$ vertically along S^1_z. A *right attractor* is an oriented cycle $\mathrm{RA}^{(k)} \subset S^{(k)}$ such that at each triple vertex, where two edges of $S^{(k)}$ go up, the cycle $\mathrm{RA}^{(k)}$ goes to the right. Denote by $(q^{(k)}, r^{(k)})$ *the winding numbers* of $\mathrm{RA}^{(k)}$ in the vertical direction S^1_z and reversed horizontal direction $(-S^1_t)$, respectively. Let $e^{(k)}\colon S^{(k)} \to \mathrm{ZT}$ be the k-th *level embedding* induced by the torus projection $\mathrm{pr}_{zt}\colon S^{(k)} \subset \mathrm{TG}(\hat{\beta}) \to \mathrm{ZT} = S^1_z \times S^1_t$, see Lemma 3.8 (b) below.

One right attractor of each $S^{(k)}$, $k = 1, 2, 3$, is shown by fat arcs in Fig. 13. In both pictures the 6 marked right attractors have the same winding numbers $(1, 0)$.

Lemma 3.8. *Let* $\mathrm{TG}(\hat{\beta}) \subset \mathbb{T}$ *be the trace graph of a closed braid* $\hat{\beta}$*, where* $\beta \in B_n$.
(a) *Any level subgraph* $S^{(k)}$ *has only trivalent vertices*; *at each vertex* 1 *edge goes down*, 2 *edges go up or vice versa with respect to the projection* $\mathrm{pr}_z : \mathrm{TG}(\hat{\beta}) \to S^1_z$.
(b) *Any level subgraph* $S^{(k)}$ *projects* 1-1 *to its image under* $\mathrm{pr}_{zt} : S^{(k)} \to \mathrm{ZT}$.
(c) *Subgraphs* $S^{(k)}$ *and* $S^{(m)}$ *have common points under* pr_{zt} *if and only if* $|k-m| = 1$; *the adjacent subgraphs can meet only in triple vertices as in* Fig. 12.
(d) *If* $k > m+1$ *then the edges of* $S^{(k)}$ *cross over* $S^{(m)}$ *under* $\mathrm{pr}_{zt} : \mathrm{TG}(\hat{\beta}) \to \mathrm{ZT}$.
(e) *Each level subgraph* $S^{(k)}$ *has at least one right attractor. Its vertical winding number* $q^{(k)}$ *is positive. Any two right attractors in* $S^{(k)}$ *have no common points.*
(f) *Under the shift* $t \mapsto t+\pi$ *each level subgraph* $S^{(k)}$ *maps to the subgraph* $S^{(n-k)}$.

Proof. (a) A triple vertex $v \in \mathrm{TG}(\hat{\beta})$ corresponds to a triple intersection (qrs) of strands from β, see Fig. 10. Let k be the level of the crossing p formed by the distant strands of q and s right below (qrs). By Lemma 3.4 (a) the crossing p is associated to a point below v in the middle trace circle passing through v. Right above p the crossings formed by the strands (qr) and (rs) have the same level k. The 3 other types of crossings have the same level $k+1$ or $k-1$, see Fig. 12.

(b) If the trace graph $\mathrm{TG}(\hat{\beta})$ has a crossing under the projection $\mathrm{pr}_{zt} : \mathrm{TG}(\hat{\beta}) \to \mathrm{ZT}$ then the points forming the crossing have the same z-coordinate and different x-coordinates. Hence they correspond to 2 crossings of some diagram $\mathrm{pr}_{xz}(\mathrm{rot}_t(\hat{\beta}))$. Definition 3.5 implies that the levels of these crossings differ at least by 2.

The items (c) and (d) follow directly from the above arguments, see Fig. 13.

(e) Starting with any vertex in $S^{(k)}$ and going always to the right in finitely many steps we will get a closed cycle oriented vertically, ie $q^{(k)} > 0$. If two right attractors in $S^{(k)}$ have a common vertex then they go along the same path and coincide.

(f) Let a point $p \in S^{(k)}$ correspond to a pair $(p_i, p_j) \in \hat{\beta}$ in a meridional disk $\mathrm{D}_{xy} \times \{z\} \subset V$. The level k is equal to 1 plus the number of intersections $\mathrm{Int}\, D \cap \hat{\beta}$, see Fig. 12. Under $t \mapsto t+\pi$, the pair (p_i, p_j) converts to (p_j, p_i), the disk D goes to the complementary disk $D' = \mathrm{D}_{xy} \times \{z\} - D$. Then the level of (p_j, p_i) is 1 plus the number of intersections $\mathrm{Int}\, D' \cap \hat{\beta}$, i.e. $1 + (n-2-(k-1)) = n-k$. □

4. Combinatorial encoding trace graphs up to isotopy

4.1. Reconstructing a closed braid from its trace graph. The moves on trace graphs are in Fig. 5 and Fig. 6. The trace graph of a closed braid in general position has the combinatorial features summarized below.

DEFINITION 4.1. An embedded finite graph $G \subset \mathbb{T}$ is a *generic* trace graph if
- under $t \mapsto t+\pi$ the graph G maps to its image under the symmetry in S^1_z;
- G splits into *trace circles* monotonic with respect to $\mathrm{pr}_z : G \to S^1_z$, they should intersect in *triple vertices* of G and verify Lemmas 3.1, 3.4;
- G splits into $n-1$ *level subgraphs* satisfying the conclusions of Lemma 3.8.

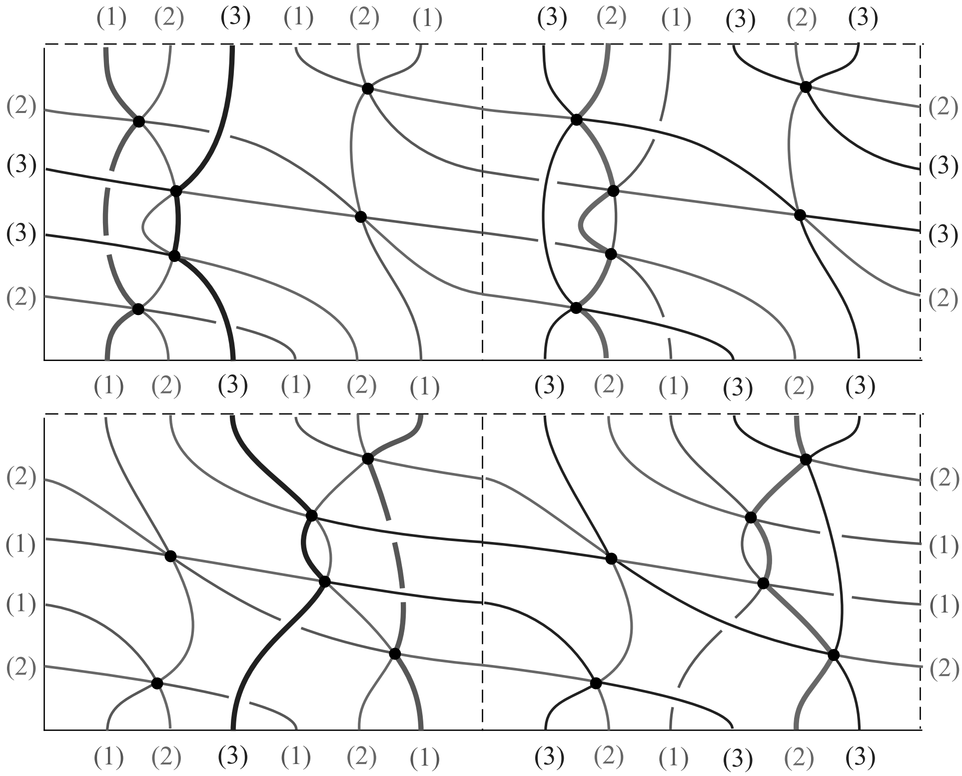


Fig. 13. Splittings of $\mathrm{TG}(\widehat{\sigma_2\sigma_3^2\sigma_2})$ and $\mathrm{TG}(\widehat{\sigma_2\sigma_1^2\sigma_2})$ into level subgraphs.

A smooth family of trace graphs $\{G_s\}$, $s \in [0, 1]$, is called *an equivalence* if

- for all but finitely many moments $s \in [0, 1]$, the trace graphs G_s are generic;
- at each critical moment, G_s changes by a trihedral or tetrahedral move.

An *isotopy* of trace graphs is an equivalence through generic trace graphs only.

Now we reconstruct a closed braid from its generic trace graph with markings.

Lemma 4.2. *For a braid $\beta \in B_n$, the closure $\hat{\beta} \subset V$ can be reconstructed up to isotopy in the solid torus from its generic trace graph G with markings.*

Proof. Consider a vertical section $P_t = G \cap (\mathrm{A}_{xz} \times \{t\})$ not containing vertices of G. Then P_t is a finite set of points with markings $(ij)[k]$, where $k \in \{1, \ldots, \gcd(n_i, n_j)\}$, see Lemma 3.1 (a). The points of P_t will play the role of crossings of a diagram of $\hat{\beta}$.

The labelled set P_t defines the *Gauss* diagram GD_t as follows. Take $\bigsqcup_{i=1}^m S_i^1$, split each oriented circle S_i^1 into n_i arcs and number them by $1, \ldots, n_i$ according to the orientation. Mark several points in the q-th arc of S^1_i in a 1-1 correspondence and the same order with the points of P_t projected under pr_z: $P_t \to S^1_z$ and having labels $(ij)[k]$ or $(ji)[k]$ for $k = 1, \dots, \gcd(n_i, n_j)$.

So each point of P_t gives 2 marked points in $\bigsqcup_{i=1}^m S^1_i$ labelled with $(ij)[k]$ and $(ji)[k]$. Connect them by a chord and get the Gauss diagram GD_t. The zero Gauss diagram GD_0 is realizable by the given diagram of the closed braid $\hat\beta$. The Gauss diagram GD_t gives rise to a diagram of a closed braid isotopic to $\hat\beta$ since the transformation from GD_0 to GD_t is clearly realizable by an isotopy of closed braids. □

Using Lemma 4.2, we state Theorem 1.4 in a slightly different form.

Proposition 4.3. *Closed braids $\hat\beta_0$ and $\hat\beta_1$ are isotopic in the solid torus V if and only if* $\mathrm{TG}(\hat\beta_0)$ *and* $\mathrm{TG}(\hat\beta_1)$ *are equivalent in the sense of* Definition 4.1.

4.2. Trace codes of trace graphs. Any curve in $\mathrm{ZT} = S^1_z \times S^1_t$ has a homology class (u, w), where u is the winding number in the vertical direction S^1_z, w is the winding number in the direction opposite to S^1_t. Take a generic trace graph G from Definition 4.1.

DEFINITION 4.4. A cycle in a level subgraph $S^{(k)} \subset G$ is called *trivial* if it bounds a disc under the embedding $e^{(k)}$: $S^{(k)} \to \mathrm{ZT}$. Any trivial cycle has an orientation induced by the oriented torus ZT. Any non-trivial cycle can be oriented in such a way that its vertical (possibly, horizontal too) winding number is non-negative.

A level subgraph is said to be *degenerate*, if all its non-trivial cycles have homology classes that are multiples of each other in $H_1(\mathrm{ZT}) = \mathbb{Z} \oplus \mathbb{Z}$. Let a level subgraph $S^{(k)}$ be non-degenerate. Denote by $(q^{(k)}, r^{(k)})$ the homology class of a right attractor. Among all non-trivial cycles in $S^{(k)}$ choose *maximal* cycles with homology classes (u, w) such that the value $M = u/q^{(k)} - w/r^{(k)}$ is non-zero and maximal.

Recall that $q^{(k)} > 0$ by Lemma 3.8 (e). If $r^{(k)} = 0$, then set $M = w$. The non-degenerate graph $S^{(k)}$ should contain non-trivial cycles with $M \neq 0$. If there are maximal cycles with different homology classes, then take one with maximal vertical number u. Now the *maximal* homology class $(u^{(k)}, w^{(k)})$ of $S(k)$ is well-defined.

By Lemma 3.1 trace circles in a trace graph are distinguished by their markings. Any right attractor can be oriented in such a way that its vertical winding number is positive. So right attractors are encoded by cyclic words of vertices.

DEFINITION 4.5. Choose a base point in each trace circle of a generic trace graph G. Enumerate all vertices of a trace circle $T_{(ab)[i]} \subset G$ by $(ab)[i]_1, (ab)[i]_2, \dots$. A triple vertex $v \in G$ can be encoded by an *ordered triplet* $\{(ab)[i]^{(k)}_x, (ac)[j]^{(k\pm1)}_y, (bc)[m]^{(k)}_z\}$. The *trace code* TC contains the following 3 *pieces* of data.

- The *first piece* consists of the ordered triplets associated to the vertices of G.

- The *second piece* contains the homology classes $(q^{(k)}, r^{(k)})$ of right attractors for each level subgraph $S^{(k)}$, $k = 1, \dots, n-1$.
- The *third piece* is the set of maximal homology classes $(u^{(k)}, w^{(k)})$ introduced for each level subgraph $S^{(k)}$ in Definition 4.4, $k = 1, \dots, n-1$.

Two trace codes are called *identical*: $\mathrm{TC}_1 = \mathrm{TC}_2$ if theirs three pieces coincide.

Our aim is to reconstruct the embedding of a generic closed trace graph G into the thickened torus $\mathbb{T}$ from its trace code $\mathrm{TC}(G)$, see Lemma 5.1. Lemma 4.6 proves this for a level subgraph $S^{(k)} \subset G$. Recall that an *isotopy* in the torus ZT is a smooth family of diffeomorphisms $F_s \colon \mathrm{ZT} \to \mathrm{ZT}$, where $s \in [0, 1]$, $F_0 = \mathrm{id}_{\mathrm{ZT}}$.

Lemma 4.6. *Let G be the trace graph of a closed braid $\hat{\beta}$, where $\beta \in B_n$.*
(a) *The embedding $e^{(k)} \colon S^{(k)} \to \mathrm{ZT}$ of a degenerate level subgraph $S^{(k)} \subset G$ can be reconstructed by its ordered triplets and the homology class $(q^{(k)}, r^{(k)})$ of its right attractor up to Dehn twists around a right attractor and isotopy in* ZT.
(b) *The embedding $e^{(k)} \colon S^{(k)} \to \mathrm{ZT}$ of a non-degenerate level subgraph can be reconstructed up to isotopy in* ZT *by its ordered triplets, the homology class $(q^{(k)}, r^{(k)})$ of its right attractor and the maximal homology class $(u^{(k)}, w^{(k)})$ of $S^{(k)}$.*

Proof. (a) A right attractor $\mathrm{RA}^{(k)} \subset S^{(k)}$ can be recognized using the set of ordered triplets of vertices. Embed $\mathrm{RA}^{(k)}$ into ZT according to its winding numbers $(q^{(k)}, r^{(k)})$. Add other vertices and edges of $S^{(k)}$ to get an embedding of the connected component of $S^{(k)}$ containing the chosen attractor. If $S^{(k)}$ is non-connected, there is another right attractor with the same homology class $(q^{(k)}, r^{(k)})$.

We repeat the above steps for all connected components of $S^{(k)}$. The image of the resulting embedding is contained in one or several annuli with the prescribed homology class $(q^{(k)}, r^{(k)})$. The whole embedding $S^{(k)} \to \mathrm{ZT}$ is well-defined up to Dehn twists around a right attractor and isotopy in ZT.

(b) For a non-degenerate subgraph $S^{(k)}$, we construct an embedding $S^{(k)} \subset \mathrm{ZT}$ as in (a). We have to improve this embedding by a suitable Dehn twist around a right attractor in such a way that the maximal homology class is $(u^{(k)}, w^{(k)})$.

The number of different homology classes is linear with respect to the number of vertices in $S^{(k)}$. We look at non-trivial cycles in the constructed embedding. Let J be the algebraic intersection number of a right attractor $\mathrm{RA}^{(k)} \subset S^{(k)}$ and a non-trivial cycle with a homology class (u, w).

The Dehn twist around $\mathrm{RA}^{(k)}$ acts on the homology: $(u, w) \mapsto (u + Jq^{(k)}, w + Jr^{(k)})$. Then $M = u/q^{(k)} - w/r^{(k)}$ is invariant under all Dehn twists around $\mathrm{RA}^{(k)}$. In the already embedded graph $S^{(k)} \subset \mathrm{ZT}$ we may recognize all non-trivial *maximal* cycles with the maximal value M computed using $(u^{(k)}, w^{(k)})$.

If there are two maximal cycles with different classes (u, w) and (u', w'), then $(u - u')/q^{(k)} = (w - w')/r^{(k)} = i$, hence $(u, w) = (u', w') + i(q^{(k)}, r^{(k)})$ for some i. Since $q^{(k)}$ and $r^{(k)}$ are coprime then i is integer. Then both cycles have the same intersection number J with the right attractor $\mathrm{RA}^{(k)}$. So a Dehn twist around $\mathrm{RA}^{(k)}$ acts on the set of the homology classes of all maximal cycles as a shift by $J(q^{(k)}, r^{(k)})$.

We know that among maximal cycles we can find one with the homology class $(u^{(k)}, w^{(k)})$, the vertical number $u^{(k)}$ is maximal possible. Let (u, w) be the homology class of a maximal cycle C with the maximal vertical number u in the embedding $S^{(k)} \subset \mathrm{ZT}$. There is an integer i such that $(u^{(k)}, w^{(k)}) - (u, w) = i\,J(q^{(k)}, r^{(k)})$.

The i Dehn twists around the right attractor $\mathrm{RA}^{(k)}$ convert the cycle C into a required cycle $\tilde{C}$ with the maximal class $(u^{(k)}, v^{(k)})$. The final embedding $S^{(k)} \subset \mathrm{ZT}$ contains a basis consisting of $\tilde{C}$ and $\mathrm{RA}^{(k)}$ with the prescribed homology classes. Therefore the embedding is well-defined up to isotopy in ZT. □

If all level subgraphs of G are non-degenerate, we may forget about levels in the trace code $\mathrm{TC}(G)$. The subgraphs $S^{(k)} \subset G$ should be connected and two adjacent subgraphs meet at each triple vertex, see Lemma 3.8 (c). Hence the levels of subgraphs can be reconstructed up to the inversion $(1, 2, \dots, n-1) \mapsto (n-1, \dots, 2, 1)$, which corresponds to the time shift $t \mapsto t + \pi$. In the second and third pieces of $\mathrm{TC}(G)$ we may leave only the homology class of a right attractor $\mathrm{RA}^{(1)}$ and the maximal homology class $(u^{(1)}, w^{(1)})$ of the first level subgraph $S^{(1)}$.

5. Recognizing trace graphs in polynomial time

5.1. Recognizing trace graphs up to isotopy.

Lemma 5.1. *Two generic trace graphs G_0 and G_1 are isotopic in the thickened torus $\mathbb{T}$ if and only if their trace codes $\mathrm{TC}(G_0)$ and $\mathrm{TC}(G_1)$ become identical after suitable cyclic permutations of vertices in trace circles.*

Proof. The part *only if* follows from the fact that the trace code is invariant under isotopy in $\mathbb{T}$. The part *if* says that the embedding of a trace graph G into the thickened torus $\mathbb{T} = \mathrm{A}_{xz} \times S^1_t$ can be reconstructed from its trace code.

By Lemma 4.6 we may reconstruct embeddings of level subgraphs $S^{(k)} \subset G$ into the torus ZT. Two embeddings of $S^{(1)}$ and $S^{(2)}$ can be joint together since the union $S^{(1)} \cup S^{(2)}$ should be embedded into ZT by Lemma 3.8 (c). The resulting embedding is well-defined up to isotopy in ZT provided that either one of the subgraphs $S^{(1)}$ and $S^{(2)}$ is non-degenerate or their right attractors have distinct homology classes.

We embed the third subgraph $S^{(3)}$ into ZT to get a joint embedding $S^{(2)} \cup S^{(3)} \subset \mathrm{ZT}$ as above. The union $S^{(1)} \cup S^{(2)} \cup S^{(3)}$ can be already considered as an embedding into the thickened torus $\mathbb{T}$ since the edges of $S^{(3)}$ should cross over $S^{(1)}$ in ZT.

The final embedding $G \subset \mathbb{T}$ is well-defined up to isotopy if either one of the subgraphs $S^{(k)}$ is non-degenerate or there are two right attractors with different homology classes. Otherwise all $S^{(k)}$ are degenerate and the embedding $G \subset \mathbb{T}$ is invariant under 3-dimensional Dehn twists around the common right attractor. □

Proposition 5.2 gives a (surprisingly) polynomial algorithm recognizing complicated topological objects: trace graphs up to isotopy in a thickened torus.

Proposition 5.2. *Let $\beta, \beta' \in B_n$ be braids of length $\leq l$. There is an algorithm of complexity $C(n/2)^{n^2/8}(6l)^{n^2-n+1}$ to decide whether $\mathrm{TG}(\hat\beta)$ and $\mathrm{TG}(\hat\beta')$ are isotopic in the thickened torus $\mathbb{T}$, where the constant C does not depend on l and n. In the case of pure braids, the power $n^2/8$ can be replaced by 1. If the closure of a braid is a knot, a single circle in the solid torus, then the complexity reduces to $Cn(6l)^{n-1}$.*

Proof. By Lemma 2.1 we may assume that the trace graphs $\mathrm{TG}(\hat\beta), \mathrm{TG}(\hat\beta')$ have $Q \leq 2l(n-2)$ triple vertices. If we fix numbers k in markings $(ij)[k]$ with $i \neq j$ and a base point in each trace circle then we can construct trace codes $\mathrm{TC}(\hat\beta), \mathrm{TC}(\hat\beta')$ of $\mathrm{TG}(\hat\beta)$, $\mathrm{TG}(\hat\beta')$, see Definition 4.5. The trace codes $\mathrm{TC}(\beta), \mathrm{TC}(\beta')$ can be compared in linear time with respect to the number Q of triple vertices.

By Lemma 3.1 (a) the graph $\mathrm{TG}(\hat\beta)$ splits into $N(\beta)$ trace circles. Denote by $k_1, \dots, k_{N(\beta)}$ the number of triple vertices in the trace circles of $\mathrm{TG}(\hat\beta)$. Then there are exactly $k_1 k_2 \cdots k_{N(\beta)}$ choices of base points in the trace cirles. Since $k_1 + \cdots + k_{N(\beta)} = 3Q \leq 6l(n-2)$, we have

$$k_1 k_2 \cdots k_{N(\beta)} \leq \left(\frac{k_1 + \cdots + k_{N(\beta)}}{N(\beta)}\right)^{N(\beta)} \leq \left(\frac{6l(n-2)}{N(\beta)}\right)^{N(\beta)} \leq (6l)^{n^2-n}$$

due to the estimates $n-1 \leq N(\beta) \leq n(n-1)$ from Lemma 3.1 (b).

Let $(n_1, \dots, n_m)$ be the lengths of cycles of the induced permutation $\tilde\beta \in S_n$. There are $\leq n/2$ *non-trivial* cycles with lengths $n_i > 1$. For each pair of non-trivial cycles with lengths (n_i, n_j), there are $\gcd(n_i, n_j)$ choices of numbers k in markings $(ij)[k]$, i.e. totally $\prod_{i<j} \gcd(n_i, n_j)$. Since the number of pairs is $\leq \binom{n/2}{2}$ and $\gcd(n_i, n_j) \leq n/2$, the number of choices $\leq (n/2)^{\binom{n/2}{2}} \leq (n/2)^{n^2/8-1}$ for $n \geq 4$.

With a fixed choice of markings and base points, we check whether $\mathrm{TC}(\hat\beta) = \mathrm{TC}(\hat\beta')$ with complexity $Cl(n-2)$. So, the final complexity of the algorithm is $C(n/2)^{n^2/8}(6l)^{n^2-n+1}$. For pure braids, markings (ij) without $[k]$ are well-defined and we may replace $n^2/8$ by 1. If $\hat\beta$ is a knot, then $N(\beta) = n-1$, markings $[k] \in \{1, \dots, n-1\}$ are well-defined and the complexity reduces to $Cn(6l)^{n-1}$. □

5.2. Recognizing trace graphs up to trihedral moves. Now we extend Proposition 5.2 to recognize trace graphs up to trihedral moves.

DEFINITION 5.3. Let G be a generic trace graph from Definition 4.1. A *trihedron* $T \subset G$ is a subgraph homeomorphic to the graph θ with 3 edges connecting 2 vertices. A trihedron in a generic trace graph G is called *embedded* if the interiors of its edges do not contain vertices of G. After eliminating (in any order) all embedded trihedra of $\mathrm{TG}(\hat\beta)$ we get a *reduced* trace graph $\overline{\mathrm{TG}}(\hat\beta)$.

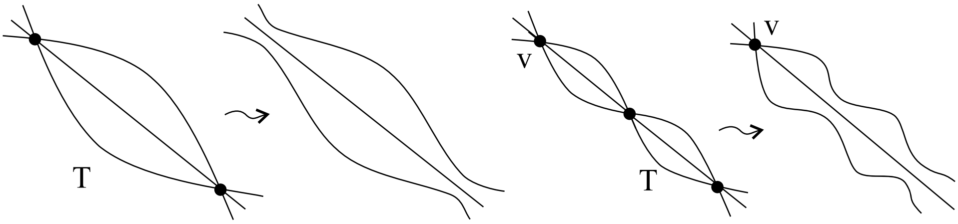


Fig. 14. Simulation of the appearance of a trihedron.

Lemma 5.4. *Let* $\overline{\mathrm{TG}}(\hat\beta), \overline{\mathrm{TG}}(\hat\beta')$ *be reduced trace graphs of closed braids* $\hat\beta, \hat\beta'$, *respectively. The original trace graphs* $\mathrm{TG}(\hat\beta), \mathrm{TG}(\hat\beta')$ *are equivalent through trihedral moves if and only if the reduced graphs* $\overline{\mathrm{TG}}(\hat\beta), \overline{\mathrm{TG}}(\hat\beta')$ *are isotopic in* $\mathbb{T}$.

Proof. The part *if* is trivial since reduced graphs are obtained by trihedral moves.

The part *only if*. The given equivalence between the original graphs provides an equivalence $\{G_s\}$ through trihedral moves only, where $s \in [0,1]$, $G_0 = \overline{\mathrm{TG}}(\hat\beta)$ and $G_1 = \overline{\mathrm{TG}}(\hat\beta')$. The trihedral moves in $\{G_s\}$ can create or delete only embedded trihedra. We simulate the creation of each trihedron T as shown in Fig. 14.

Either T will disappear completely by a further trihedral move in $\{G_s\}$ or an adjacent trihedron will be deleted and will destroy T. In both cases we miss the deleting move in the simulation. After simulating all trihedral moves the equivalence $\{G_s\}$ becomes a required isotopy between reduced graphs. □

Proof of Theorem 1.5. Embedded trihedra in a trace graph can be recognized in quadratic time with respect to the number of vertices. For each pair of vertices, we check if they are connected by three edges not containing other vertices. After that the algorithm of Proposition 5.2 can be applied to the reduced trace graphs $\overline{\mathrm{TG}}(\hat\beta), \overline{\mathrm{TG}}(\hat\beta')$ and gives the required polynomial complexity in the braid length. □

A *meridional quadrisecant* of a closed braid $\hat\beta$ in the solid torus V is a straight line in a meridional disk $\mathrm{D}_{xy} \times \{z\}$ meeting $\hat\beta$ in 4 points. For an equivalence $\{\hat\beta_s\}$ without meridional quadrisecants, the canonical loops of rotated braids $\mathrm{rot}_t(\hat\beta_s)$ can pass only through $\Sigma_{\asymp}$, $\Sigma_{\curlyvee}$ and can touch $\Sigma_{\divideontimes}$, see Subsection 2.1. Passing through $\Sigma_{\ast}$ creates a a meridional quadrisecant in a closed braid. Passing through a tangency with $\Sigma_{\divideontimes}$ corresponds to a trihedral move in Fig. 5.

Corollary 5.5. *Let* $\beta, \beta' \in B_n$ *be braids of length* $\le l$. *There is an algorithm of complexity* $C(n/2)^{n^2/8}(6l)^{n^2-n+1}$ *to decide whether there is an equivalence* $\{\hat\beta_s\}$ *such that* $\mathrm{CL}(\hat\beta_s)$ *can pass only through* $\Sigma_{\asymp}$, $\Sigma_{\curlyvee}$ *and can touch* $\Sigma_{\divideontimes}$ *for* $s \in [0,1]$.

Proof. The closed braids $\hat{\beta}, \hat{\beta}'$ are equivalent in the above sense if and only if their trace graphs $\mathrm{TG}(\hat{\beta}), \mathrm{TG}(\hat{\beta}')$ are equivalent through trihedral moves only. So the algorithm of Theorem 1.5 can be applied to $\mathrm{TG}(\hat{\beta}), \mathrm{TG}(\hat{\beta}')$. □

The following conjecture implies that, for closed braids having trace graphs without trihedra, the conjugacy problem can be solved in a polynomial time similarly to Theorem 1.5 since tetrahedral moves do not change the number of triple vertices.

Conjecture 5.6. *If trace graphs of isotopic closed braids have no trihedra then they are related by tetrahedral moves only.*

The idea is to simplify an equivalence of trace graphs cancelling moves that create and remove trihedra. The ultimate aim is to extend Conjecture 5.6 to all closed braids making an equivalence of trace graphs monotone with respect to the number of triple vertices, which would give a polynomial algorithm for all braids.

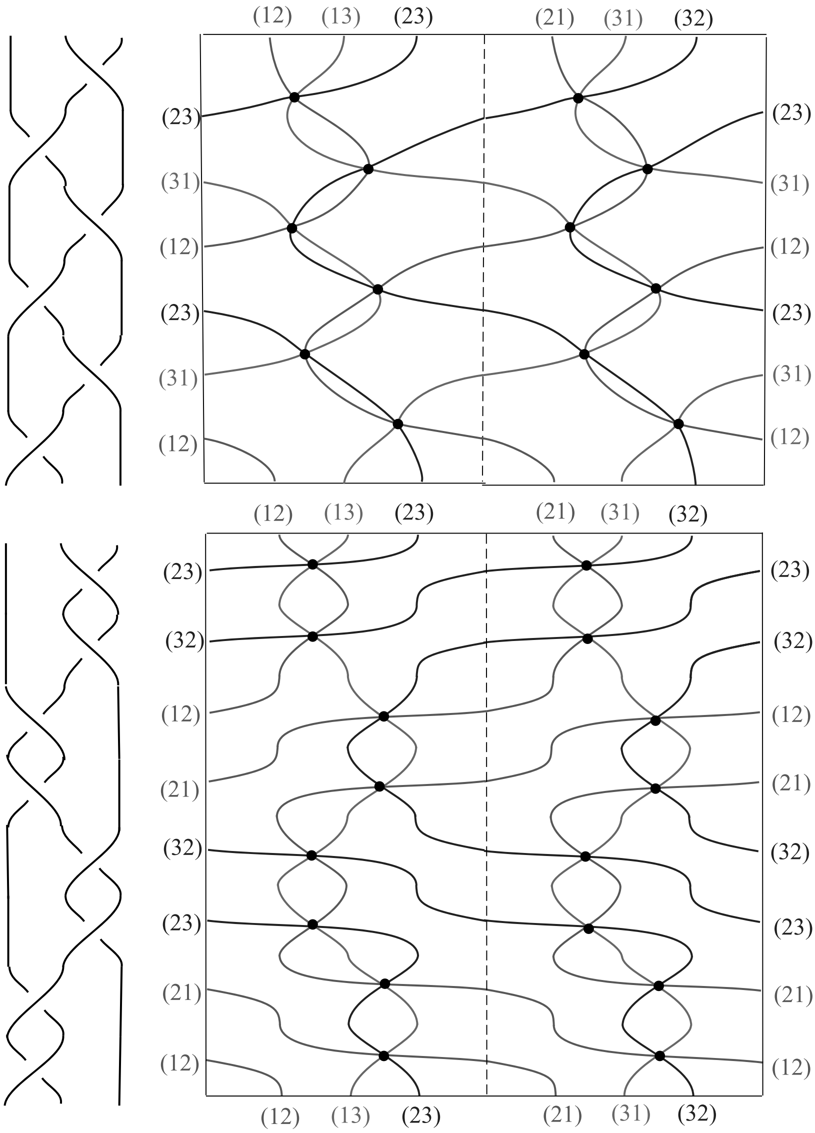
6. A geometric recognizing 3-braids up to conjugacy

According to González-Meneses [11], if two braids α and β satisfy $\alpha^k = \beta^k$ in B_n for some $k \neq 0$, then α and β are conjugate. It follows that braids α and β are conjugate if and only if α^k and β^k are conjugate for some $k \neq 0$, see González-Meneses [11, Corollary 1.2]. For any braid $\beta \in B_n$, there is a power k such that the permutation $\tilde{\beta}^k \in S_n$ induced by β^k is trivial, hence β^k is pure. So the conjugacy problem for the braid group B_n reduces to the case of pure braids.

6.1. Cyclic invariants based on 3-subbraids. In this subsection we recognize closed pure 3-braids up to isotopy in the solid torus by using invariants of their trace graphs calculable in a linear time with respect to the braid length. Then trace circles in the trace graph $\mathrm{TG}(\hat{\beta})$ can be denoted simply by $T_{(ij)}$, where $i, j \in \{1, \ldots, n\}$. We shall define cyclic invariants depending on 3-subbraids of β and distinguishing all pure 3-braids up to conjugacy.

Take a pure braid $\beta \in B_n$ and enumerate the components of $\hat{\beta}$ by $1, \ldots, n$. Fix three pairwise disjoint indices $i, j, k \in \{1, \ldots, n\}$. We shall define the cyclic invariants $C_{(ij)}$ depending on the 3-subbraid β_{ijk} based on the strands i, j, k.

DEFINITION 6.1. Take the reduced trace graph $\overline{\mathrm{TG}}(\hat{\beta}_{ijk})$ well-defined up to isotopy of β_{ijk} by Lemma 5.4 since tetrahedral moves are not applicable for 3-braids. For each triple vertex $v \in T_{(ij)}$, we write the ordered triplet of the markings of trace circles passing through v in the order from left to right below v, see Fig. 16. The vertices and their triplets are ordered vertically in the direction S^1_z. Then $C_{(ij)}(\hat{\beta})$ is a vertical column of triplets, the invariant is defined up to cyclic permutation, see Fig. 15. Similarly we define $C_{(ik)}(\hat{\beta})$, $C_{(jk)}(\hat{\beta})$.



$$C_{(12)}=\begin{cases}(23)(13)(12)\\(32)(12)(13)\\(12)(32)(31)\\(12)(13)(23)\\(13)(12)(32)\\(31)(32)(12)\end{cases}\quad C_{(13)}=\begin{cases}(23)(13)(12)\\(32)(12)(13)\\(21)(23)(13)\\(12)(13)(23)\\(13)(12)(32)\\(13)(23)(21)\end{cases}\quad C_{(23)}=\begin{cases}(23)(13)(12)\\(23)(21)(31)\\(21)(23)(13)\\(12)(13)(23)\\(31)(21)(23)\\(13)(23)(21)\end{cases}$$

Fig. 15. Trace graphs of the Borromean links: $(\sigma_1\sigma_2^{-1})^3$ and $\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^{-2}$.

Due to the symmetry of $\mathrm{TG}(\hat{\beta})$ under the shift $t \mapsto t + \pi$, the other invariants $C_{(ji)}, C_{(ki)}, C_{(kj)}$ can be reconstructed from the already defined ones.

EXAMPLE 6.2. Fig. 15 contains the trace graphs of the closures of the 3-braids $(\sigma_1\sigma_2^{-1})^3$ and $\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^{-2}$. Both closures are Borromean links, i.e. the braids are conjugate. In fact the second graph is isotopic to the first one by eliminating the couple of embedded trihedra. The cyclic invariants $C_{(12)}, C_{(13)}, C_{(23)}$ are shown below the pictures. The vertices of the embedded trihedra in the second trace graph are encoded by (12)(13)(23) and (23)(13)(12), i.e. the extreme markings swap their positions. Moreover, the cyclic invariants show that the braids are not trivial.

6.2. Recognizing 3-braids up to conjugacy in a linear time.

Lemma 6.3. *Number components of two closed pure* 3*-braids* β, β' *by* 1, 2, 3. *Suppose that the ordered links* $\hat{\beta}, \hat{\beta}'$ *are isotopic in the solid torus* V. *Then the cyclic invariants* $C_{(ij)}(\hat{\beta})$ *and* $C_{(ij)}(\hat{\beta}')$ *coincide for all disjoint* $i, j \in \{1, 2, 3\}$. *The invariant* $C_{(ij)}(\hat{\beta})$ *is calculable in linear time with respect to the length of* β.

Proof. By Proposition 4.3 the trace graphs of isotopic closed braids are connected by an isotopy in the thickened torus $\mathbb{T}$, trihedral moves and tetrahedral moves. The cyclic invariants are not changed under isotopy of trace graphs. Tetrahedral moves are not applicable for 3-braids. Trihedral moves create trihedra that are recognizable by cyclic invariants and deleted in the construction of Definition 6.1. To compute $C_{(ij)}(\hat{\beta})$ we need to look at all triple vertices of the trace circle $T_{(ij)}$. The total number of vertices is not more than $2l$ by Lemma 2.1. □

Recall that closed pure 2-braids are classified up to conjugacy by the linking number lk_{12} of closed strands 1 and 2. Proposition 6.4 implies that 3-braids can be recognized up to conjugacy in linear time with respect to their length.

Proposition 6.4. *Fix closed pure* 3*-braids* β, β' *with ordered components. The braids* β, β' *are conjugate if and only if the linking numbers* $\mathrm{lk}_{12}(\hat{\beta}) = \mathrm{lk}_{12}(\hat{\beta}')$ *and the cyclic invariants* $C_{(12)}(\hat{\beta}), C_{(12)}(\hat{\beta}')$ *coincide up to cyclic permutation.*

Proof. The part *only if* is Lemma 6.3. The part *if* says the original 3-braid can be reconstructed from its invariants lk_{12} and $C_{(12)}$. Simply assume that $\mathrm{lk}_{12} = 0$, e.g. strands 1, 2 are straight, i.e. multiply both braids by $\Delta^{-2\,\mathrm{lk}_{12}}$, where $\Delta = (\sigma_1\sigma_2)^3$.

Consider a meridional disk $D_z = \mathrm{D}_{xy} \times \{z\}$ in the solid torus V, where the closed braid $\hat{\beta}$ lives. Mark the intersection points $D_z \cap \hat{\beta}$ by 1, 2, 3 according to the components of $\hat{\beta}$. Since points 1, 2 do not move in D_z while z varies, we need to know only how point 3 moves through the line connecting points 1, 2. There are 6 cases that are distiguished by ordered triplets from $C_{(12)}$.

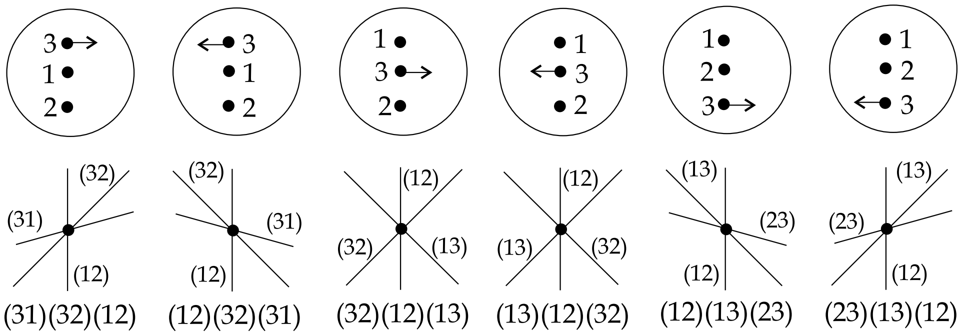


Fig. 16. Dynamic interpretation of triple vertices.

In Fig. 16 the arrow at point 3 shows its meridional velocity while z increases. In the left hand side picture, the line connecting points 1, 3 is going to have a positive slope in D_z, hence the trace circle $T_{(31)}$ is increasing as a function $z(t)$. So triplets describe neighbourhoods of associated triple vertices, which can be joined together to get a complete trace graph leading to a braid by Lemma 4.2. □

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