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Compressed Drinfeld associators

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To my mother

Abstract

Drinfeld associator is a key tool in computing the Kontsevich integral of knots. A Drinfeld associator is a series in two non-commuting variables, satisfying highly complicated algebraic equations—hexagon and pentagon. The logarithm of a Drinfeld associator lives in the Lie algebra L generated by the symbols a, b, c modulo $[a, b] = [b, c] = [c, a]$. The main result is a description of compressed associators that obey the compressed pentagon and hexagon in the quotient $L/[[L, L], [L, L]]$. The key ingredient is an explicit form of Campbell–Baker–Hausdorff formula in the case when all commutators commute.

Keywords: Drinfeld associator; Compressed associator; Kontsevich integral; Zeta function; Knot; Hexagon equation; Pentagon equation; Bernoulli numbers; Extended Bernoulli numbers; Campbell–Baker–Hausdorff formula; Lie algebra; Chord diagrams; Vassiliev invariants; Compressed Vassiliev invariants

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1. Introduction

1.1. Motivation and the previous results

Let A be a quasi-Hopf algebra [9] with a non-commutative non-associative coproduct Δ. Roughly speaking, *an associator* is an element $\Phi \in A^{\otimes 3}$ controlling non-associativity of the coproduct Δ. Another element $R \in A^{\otimes 2}$ characterizes non-commutativity of Δ. V. Drinfeld found the universal Knizhnik–Zamolodchikov associator $(R_{\mathrm{KZ}}, \Phi_{\mathrm{KZ}})$ by using analytic methods—differential equations and iterated integrals. Also Drinfeld proved that there is an iterative algebraic procedure for finding a universal formula of an associator over the rationals. Although this procedure is constructive, it does not give rise to a closed explicit formula.

The main motivation is the construction of the Kontsevich integral of knots via associators, investigated by T.Q.T. Le, J. Murakami [16], and D. Bar-Natan [3]. Other combinatorial constructions of the universal Vassiliev invariant are in [7,20]. Recall that the Kontsevich integral takes values in the algebra A of chord diagrams. The LM–BN construction gives an isotopy invariant of parenthesized framed tangles [4] expressed via a Drinfeld associator that is a solution of rather complicated equations—hexagon and pentagon (the same as mentioned above). Any solution of these equations gives rise to a knot invariant. Le and Murakami [15] have proven that the resulting invariant is independent of a particularly chosen associator and coincides with the Kontsevich integral from [13] provided $R = \exp(t^{12})$. In other words, if one knew all coefficients of at least one associator, then one could calculate the whole Kontsevich integral for any knot. Another approach of Bar-Natan, Le, and D. Thurston has led to a formula for the Kontsevich integral of the unknot and all torus knots in the space of Jacobi diagrams [5].

A special associator was expressed via multiple zeta values [14], i.e. via transcendental numbers. Drinfeld computed the logarithm of the same associator in the case when all commutators commute by using classical zeta values [10]. This result was the starting point of the present researches. Bar-Natan calculated a rational Drinfeld associator up to degree 7 in [3]. J. Lieberum [18] determined explicitly a rational associator in a completion of the universal enveloping algebra of the Lie superalgebra $\mathrm{gl}(1|1)^{\otimes 3}$. Up to now a closed formula of a rational associator is still unknown [19, p. 433, Problem 3.13].

Extreme coefficients of all Drinfeld associators will be calculated in Theorem 1.5(c). It turns out that they are rational and expressed via the classical Bernoulli numbers B_n.

1.2. Basic definitions

Definition 1.1 (associative algebra A_n, algebra of chord diagrams $A(X)$).

(a) For each $n \geqslant 2$, let the associative algebra A_n over the field $\mathbb{C}$ be generated by the symbols $t^{ij} = t^{ji}$, $1 \leqslant i \neq j \leqslant n$, modulo the relations

$$[t^{ij}, t^{kl}] = 0 \quad \text{if } i, j, k, l \text{ are pairwise disjoint},$$

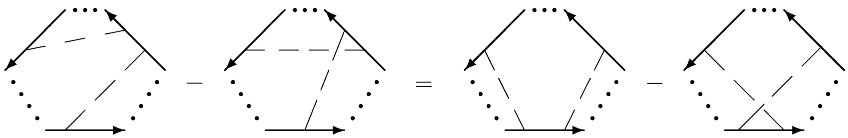
$$[t^{ij}, t^{jk} + t^{ki}] = 0 \quad \text{if } i, j, k \text{ are pairwise disjoint},$$

where *the bracket* $[\cdot,\cdot]: A_n \oplus A_n \to A_n$ is defined by $[a,b] := ab - ba$. Observe that the relations $[t^{ij}, t^{jk} + t^{ki}] = 0$ of A_n are equivalent to

$$\left[t^{ij}, t^{jk}\right] = \left[t^{jk}, t^{ki}\right] = \left[t^{ki}, t^{ij}\right] \quad \text{for all pairwise disjoint } i, j, k \in \{1, \dots, n\}.$$

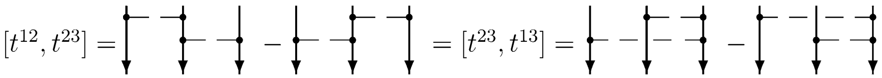
The associative algebra A_n is graded by *the degree*: $\deg(t^{ij}) = 1$.

(b) Let us define the same object A_n geometrically. Let X be a 1-dimensional oriented compact manifold, possibly non-connected and with boundary. *A chord diagram on* X is a collection of non-oriented dashed lines (*chords*) with endpoints on X. Let $A(X)$ be the linear space generated by all chord diagrams on X modulo *the* $4T$ *relations*:



The dotted arcs represent parts of the diagrams that are not shown in the figure. These parts are assumed to be the same in all four diagrams.

If $X = X_n$ is the disjoint union of n oriented segments (*strands*), then $A(X_n)$ can be endowed with a natural product. If in the definition of $A(X_n)$ one allows only horizontal chords with endpoints on n vertical strands, then the resulting algebra $A^{\text{hor}}(X_n)$ is isomorphic to the algebra A_n. Indeed, thinking of t^{ij} as a horizontal chord connecting the ith and jth vertical strands, the relations between the t^{ij} become the $4T$ relations:



Definition 1.2 (Lie algebra L_n, quotient $\overline{L}_n$, long commutators $[a_1 \dots a_k]$).

(a) The Lie algebra L_n is generated by the same generators and relations as the associative algebra A_n of Definition 1.1. The Lie algebra L_n is graded with respect to $\deg(t^{ij}) = 1$.

(b) Denote by $[L_n, L_n]$ the Lie subalgebra of L_n, generated by all commutators $[a,b]$ with $a, b \in L_n$. Introduce *the compressed quotient* $\overline{L}_n = L_n / [[L_n, L_n], [L_n, L_n]]$. Let $\widehat{A}_n$, $\widehat{L}_n$, and $\widehat{\overline{L}}_n$ be the algebras of formal series of elements from A_n, L_n, and $\overline{L}_n$, respectively.

(c) For elements $a_1, \dots, a_n$ of a Lie algebra L, set $[a_1 a_2 \dots a_k] = [a_1, [a_2, [\dots, a_k] \dots]]$.

For example, the algebras $\widehat{A}_2$ and $\widehat{L}_3$ contain the series $\exp(t^{12})$ and $\sum_{k=1}^{\infty} [(t^{12})^k t^{23}]$, respectively.

Definition 1.3 (operators Δ_k and ε_k, Drinfeld associators and compressed associators).

(a) Let t^{ij} be the generators of A_n. Let $\Delta_k : A_n \to A_{n+1}$ for $0 \leqslant k \leqslant n+1$ and $\varepsilon_k : A_n \to A_{n-1}$ for $1 \leqslant k \leqslant n$ be the algebra morphisms defined by their action on t^{ij} (here $i < j$):

$$\Delta_k\bigl(t^{ij}\bigr)=\begin{cases} t^{ij}, & \text{if } i<j<k,\\ t^{i,j+1}, & \text{if } i<k<j,\\ t^{i+1,j+1}, & \text{if } k<i<j,\\ t^{ij}+t^{i,j+1}, & \text{if } i<j=k,\\ t^{i,j+1}+t^{i+1,j+1}, & \text{if } i=k<j;\end{cases}$$

$$\varepsilon_k\bigl(t^{ij}\bigr)=\begin{cases} t^{ij}, & \text{if } i<j<k,\\ t^{i,j-1}, & \text{if } i<k<j,\\ t^{i-1,j-1}, & \text{if } k<i<j,\\ 0, & \text{if } i<j=k,\\ 0, & \text{if } i=k<j.\end{cases}$$

Δ_0 (Δ_{n+1}) acts by adding a strand on the left (right), Δ_k for $1 \leqslant k \leqslant n$ acts by doubling the kth strand and summing up all the possible ways of lifting the chords that were connected to the kth strand to the two daughter strands. The operator ε_k acts by deleting the kth strand and mapping the chord diagram to 0, if any chord in it was connected to the kth strand.

(b) *A horizontal Drinfeld associator* (briefly, *a Drinfeld associator*) is an element $\Phi \in \widehat{A}_3$ satisfying the following equations (here set $\Phi^{ijk} := \Phi(t^{ij}, t^{jk})$ and $\Phi := \Phi^{123}$)

(symmetry) $\quad \Phi \cdot \Phi^{321} = 1 \quad \text{in } \widehat{A}_3,$ (1.3a)

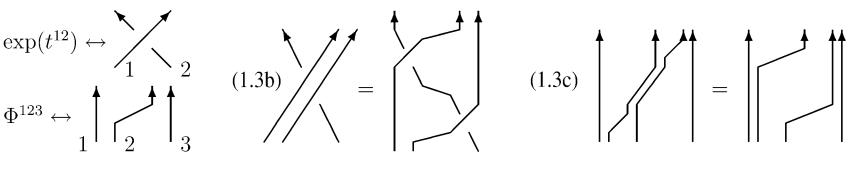
(hexagon) $\quad \Delta_1\bigl(\exp\bigl(t^{12}\bigr)\bigr) = \Phi^{312} \cdot \exp\bigl(t^{13}\bigr) \cdot \bigl(\Phi^{-1}\bigr)^{132} \cdot \exp\bigl(t^{23}\bigr) \cdot \Phi^{123}$ in $\widehat{A}_3$, (1.3b)

(pentagon) $\quad \Delta_0(\Phi) \cdot \Delta_2(\Phi) \cdot \Delta_4(\Phi) = \Delta_3(\Phi) \cdot \Delta_1(\Phi) \quad \text{in } \widehat{A}_4,$ (1.3c)

(non-degeneracy) $\quad \varepsilon_1\Phi = \varepsilon_2\Phi = \varepsilon_3\Phi = 1 \quad \text{in } \widehat{A}_2,$ (1.3d)

(group-like) $\quad \Phi = \exp(\varphi) \quad$ in $\widehat{A}_3$ for some element $\varphi \in \widehat{L}_3$. (1.3e)

A geometric interpretation of the hexagon and pentagon is shown below:



(c) If an associator $\Phi \in \widehat{A}_3$ vanishes in all odd degrees, then Φ is said to be *even*. Note that the symmetry (1.3a) implies $\varphi(a,b) = -\varphi(b,a)$ in $\widehat{L}_3$. By taking the logarithms of (1.3b) and (1.3c) and projecting them under $\widehat{L}_3 \to \widehat{\widehat{L}}_3$ and $\widehat{L}_4 \to \widehat{\widehat{L}}_4$ one gets *the compressed hexagon* $\overline{(1.3\text{b})}$ *and pentagon* $\overline{(1.3\text{c})}$, respectively. *A compressed associator* $\bar{\varphi} \in \widehat{\widehat{L}}_3$ is a solution of $\overline{(1.3\text{b})}$ and $\overline{(1.3\text{c})}$, satisfying $\bar{\varphi}(a,b) = -\bar{\varphi}(b,a)$ and $\bar{\varphi}(a,0) = \bar{\varphi}(0,b) = 0$.

Definition (1.3b) uses a non-classical normalization. Drinfeld considered the two hexagons [9]:

$$\Delta_1\left(\exp\left(\pm\frac{t^{12}}{2}\right)\right) = \Phi^{312}\cdot\exp\left(\pm\frac{t^{13}}{2}\right)\cdot\left(\Phi^{-1}\right)^{132}\cdot\exp\left(\pm\frac{t^{23}}{2}\right)\cdot\Phi^{123}.$$

To avoid huge denominators in future the change of the variables $t^{ij}\mapsto 2t^{ij}$ was made. Moreover, Bar-Natan has proven that both hexagons above are equivalent to the positive hexagon (with the sign "+") and the symmetry (1.3a), see [3, Proposition 3.7]. The logarithm $\varphi=\log(\Phi)$ of any Drinfeld associator projects under $\widehat{L}_3\to\widehat{\widehat{L}}_3$ onto a compressed associator.

Example 1.4. Bar-Natan calculated an even Drinfeld associator up to degree 7:

$$\begin{aligned}\varphi^{\mathrm{B}}(a,b) = {}&\left(\frac{[ab]}{12}-\frac{8[a^3b]+[abab]}{720}\right.\\&\left.+\frac{96[a^5b]+4[a^3bab]+65[a^2b^2ab]+68[aba^3b]+4[(ab)^3]}{90720}\right)\\&-(\text{interchange of } a\leftrightarrow b),\end{aligned}$$

where $a=t^{12}$, $b=t^{23}$. By the above normalization one needs to divide the denominators at all terms of the degree n by 2^n. Degree 7 is the maximal achievement of Bar-Natan's computer programme. Then in $\widehat{\widehat{L}}_3$ one gets:

$$\begin{aligned}\bar{\varphi}^{\mathrm{B}}(a,b) = {}&\frac{[ab]}{6}-\frac{4[a^3b]+[abab]+4[b^2ab]}{360}+\frac{[a^5b]+[b^4ab]}{945}+\frac{[a^3bab]+[ab^3ab]}{1260}\\&+\frac{23[a^2b^2ab]}{30240}+\cdots.\end{aligned}\tag{1.4}$$

1.3. Main results

For a series $f(\lambda,\mu)$, let us introduce its *even* and *odd* parts:

$$Even\big(f(\lambda,\mu)\big)=\frac{f(\lambda,\mu)+f(-\lambda,-\mu)}{2},\qquad Odd\big(f(\lambda,\mu)\big)=\frac{f(\lambda,\mu)-f(-\lambda,-\mu)}{2}.$$

Theorem 1.5 describes all compressed Drinfeld associators.

Theorem 1.5.

(a) *Any compressed Drinfeld associator* $\bar{\varphi}\in\widehat{\widehat{L}}_3$ *from Definition* 1.3(c) *is*

$$\bar{\varphi}(a,b)=\sum_{k,l\geqslant 0}\alpha_{kl}\left[a^k b^l ab\right],\quad \textit{where } a=t^{12},\ b=t^{23},\ \alpha_{kl}\in\mathbb{C}.\tag{1.5a}$$

Moreover, the coefficients α_{kl} *are symmetric*: $\alpha_{kl} = \alpha_{lk}$ *for all* $k, l \geqslant 0$.

(b) *Let* $f(\lambda, \mu) = \sum_{k,l \geqslant 0} \alpha_{kl} \lambda^k \mu^l$ *be the generating function of the coefficients* $\alpha_{kl} = \alpha_{lk}$. *Then the compressed hexagon* $\overline{(1.3b)}$ *from Definition* 1.3(c) *is equivalent to the equation*

$$f(\lambda, \mu) + e^{\mu} f(\mu, -\lambda - \mu) + e^{-\lambda} f(\lambda, -\lambda - \mu) = \frac{1}{\lambda + \mu}\left(\frac{e^{\mu} - 1}{\mu} + \frac{e^{-\lambda} - 1}{\lambda}\right). \tag{1.5b}$$

(c) *The general solution of* (1.5b) *is*

$$f(\lambda, \mu) = Even\big(f(\lambda, \mu)\big) + Odd\big(f(\lambda, \mu)\big),$$

where

$$\begin{cases} 1 + \lambda\mu \cdot Even\big(f(\lambda, \mu)\big) = \dfrac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda + \mu)}\left(\dfrac{2\omega}{e^{\omega} - e^{-\omega}} + \displaystyle\sum_{n=3}^{\infty} h_n(\lambda, \mu)\right), \\ Odd\big(f(\lambda, \mu)\big) = \dfrac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2}\left(\displaystyle\sum_{n=0}^{\infty} \tilde{\beta}_{n0}\omega^{2n} + \sum_{n=3}^{\infty} \tilde{h}_n(\lambda, \mu)\right), \end{cases} \tag{1.5c}$$

$$h_n(\lambda, \mu) = \sum_{k=1}^{[n/3]} \beta_{nk} \lambda^{2k} \mu^{2k} (\lambda + \mu)^{2k} \omega^{2n-6k} \quad \text{for } n \geqslant 3,$$

$$\beta_{nk} \in \mathbb{C}, \quad \omega = \sqrt{\lambda^2 + \lambda\mu + \mu^2}.$$

Moreover, the compressed pentagon $\overline{(1.3c)}$ *for* $\bar{\varphi} \in \widehat{\overline{L}}_3$ *follows from the symmetry* $\alpha_{kl} = \alpha_{lk}$. *In particular, any honest Drinfeld associator has the extreme coefficients*

$$\alpha_{2k,0} = \frac{2^{2k+1} B_{2k+2}}{(2k+2)!}$$

for every $k \geqslant 0$, *where* B_n *are the Bernoulli numbers. The polynomials* $\tilde{h}_n(\lambda, \mu)$ *are defined by the same formula as* $h_n(\lambda, \mu)$, *except the coefficients* $\tilde{\beta}_{nk} \in \mathbb{C}$ *are substituted for* β_{nk}. *The coefficients* $\tilde{\beta}_{n0}$ (*for* $n \geqslant 0$), β_{nk}, *and* $\tilde{\beta}_{nk}$ *are free parameters for* $1 \leqslant k \leqslant [n/3]$, $n \geqslant 3$.

By Theorem 1.5(c) the differences of all compressed associators form a linear space generated by $\beta_{nk}, \tilde{\beta}_{nk}$. The projection of any Drinfeld associator is in (1.5c), see Problem 6.10(a).

Corollary 1.6. (a, b) *There are two distinguished even compressed Drinfeld associators*

$$\bar{\varphi} = \sum_{k,l\geqslant 0} \alpha_{kl}\left[a^k b^l ab\right] \in \widehat{\overline{L}}_3$$

defined by the generating function of the coeficients α_{kl}

$$f(\lambda,\mu) = \sum_{k,l\geqslant 0} \alpha_{kl}\lambda^k\mu^l$$

as follows:

$$\textit{the first series}:\quad 1+\lambda\mu f^{\mathrm{I}}(\lambda,\mu) = \frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}\cdot\frac{2\omega}{e^{\omega}-e^{-\omega}},\quad \omega=\sqrt{\lambda^2+\lambda\mu+\mu^2}; \tag{1.6a}$$

$$\textit{the second series}:\quad 1+2\lambda\mu f^{\mathrm{II}}(\lambda,\mu) = \frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}\left(\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}+\frac{2\mu}{e^{\mu}-e^{-\mu}}-1\right). \tag{1.6b}$$

(c) *There are compressed Drinfeld associators* $\bar{\varphi} \in \widehat{\overline{L}}_3$ *defined by the Drinfeld series* $f^{\mathrm{D}}(\lambda,\mu)$:

$$1+\lambda\mu f^{\mathrm{D}}(\lambda,\mu) = \exp\left(\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}\cdot\frac{\lambda^n+\mu^n-(\lambda+\mu)^n}{(\pi\sqrt{-1})^n}\right),$$

where odd zeta values $\zeta(2n+1)$ *are considered as free parameters, see Definition* 6.1. *In particular, one gets the third distinguished compressed Drinfeld associator*:

$$\textit{the third series}:\quad 1+\lambda\mu f^{\mathrm{III}}(\lambda,\mu) = \exp\left(\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{4n(2n)!}\left((\lambda+\mu)^{2n}-\lambda^{2n}-\mu^{2n}\right)\right), \tag{1.6c}$$

where B_{2n} *are the classical Bernoulli numbers, see Definition* 2.1(b).

1.4. Scheme of proofs

Key points of proofs are listed below.

First key point: a behavior of the Bernoulli numbers ($B_n = 0$ for each odd $n \geqslant 3$).
Second key point: the Bernoulli numbers B_n can be extended in a natural way, this extension gives a compressed variant of CBH formula (Definition 2.4 and Proposition 2.8).

Third key point: properties of the extended Bernoulli numbers C_{mn} and their generating function $C(\lambda,\mu)$: a non-trivial symmetry $C(\lambda,\mu)=C(-\mu,-\lambda)$ (Lemma 2.10) and an explicit expression of $C(\lambda,\mu)$ (Proposition 2.12).

4th key point: the original hexagon equation (1.3b) can be simplified in such a way that it remains to apply CBH formula in an essential way exactly once (Lemma 3.1).

5th key point: the quotient $\overline{L}_3 = L_3/[[L_3,L_3],[L_3,L_3]]$, where a compressed associator lives, is isomorphic to a Lie algebra with a small basis of commutators (Proposition 3.4).

6th key point: the compressed hexagon equation $\overline{(1.3b)}$ is equivalent to a recursive linear system for the coefficients α_{kl} (Proposition 3.9 and Lemma 4.1).

7th key point: the extreme coefficients $\alpha_{2k,0}$ of the exact logarithm of any Drinfeld associator (not only the compressed one) are expressed via the Bernoulli numbers B_{2n} (Lemma 4.2).

8th key point: for any compressed associator, the compressed hexagon $\overline{(1.3b)}$ can be split into two equations for the even and odd parts of this associator (Lemma 4.5).

9th key point: accurate to a certain factor the general solution of the compressed hexagon $\overline{(1.3b)}$ is a series $h(\lambda,\mu)$ with the symmetry $h(\lambda,\mu)=h(\lambda,-\lambda-\mu)$ (Lemmas 4.6 and 4.12).

10th key point: non-uniqueness of compressed associators is closely related with non-uniqueness of associator polynomials (Definition 4.7 and Lemma 4.9).

11th key point: all associator polynomials can be described explicitly: in each degree $2n$ the family of all associator polynomials depends on $[n/3]$ free parameters (Proposition 4.10).

12th key point: for any compressed associator, the compressed pentagon equation $\overline{(1.3c)}$ follows from the symmetry condition $\alpha_{kl}=\alpha_{lk}$ (Proposition 5.10).

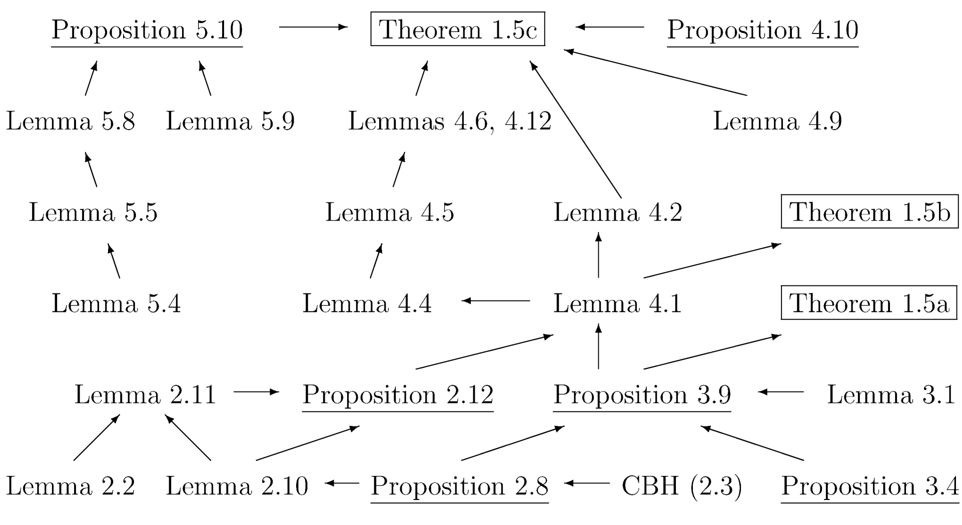


Fig. 1.

The paper is organized as follows. In Section 2 the extended Bernoulli numbers C_{mn} are introduced, one deduces a compressed variant of CBH formula in the case when all commutators commute with each other. In Sections 3 and 4 the compressed hexagon is written explicitly and solved. Section 5 is devoted to checking the compressed pentagon. Theorem 1.5(a) is proved in Section 3.2. Theorem 1.5(b) and Corollary 1.6 are checked in Sections 4.1 and 6.2, respectively. The hexagon and pentagon parts of Theorem 1.5(c) are verified in Sections 4.3 and 5.3, respectively. In Section 6.3 open problems and suggestions for future researches are formulated. Appendix contains a lot of explicit formulae discussed in the paper in their general forms.

In Fig. 1 see the scheme for the proof of Theorem 1.5. Important steps are called propositions, they are of independent interest, especially Propositions 2.8 and 2.12 together.

2. Campbell–Baker–Hausdorff formula (CBH)

This section is devoted to an explicit form of CBH formula in the case when all commutators commute with each other, see Propositions 2.8 and 2.12.

2.1. Classical recursive CBH formula

Recall a classical recursive CBH formula (Theorem 2.3) originally proved by J. Campbell [6], H. Baker [1], and F. Hausdorff [11].

Definition 2.1 (Hausdorff series H, Bernoulli numbers B_n, derivative $D = H_1 \frac{\partial}{\partial Q}$).

(a) Let L be the free Lie algebra generated by the symbols P, Q. By $\widehat{L}$ denote the algebra of formal series of elements from L. *The Hausdorff series* is $H = \log(\exp(P) \cdot \exp(Q)) \in \widehat{L}$.
(b) *The Bernoulli numbers B_n* are defined by the generating function:

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1}.$$

(c) *A derivative* of a Lie algebra L is a linear function $D : \widehat{L} \to \widehat{L}$ such that $D[x, y] = [Dx, y] + [x, Dy]$ for all $x, y \in \widehat{L}$. For an element $H_1 \in \widehat{L}$, denote by $D = H_1 \frac{\partial}{\partial Q}$ the derivative of L, which maps P onto 0 and Q onto H_1.

One can verify that $B_0 = 1$, $B_1 = -\frac{1}{2}$ and that $\frac{x}{e^x-1} + \frac{x}{2}$ is an even function, which shows that $B_n = 0$ for odd $n \geqslant 3$. *The first key point*: the function $\frac{x}{e^x-1}$ vanishes in almost all odd degrees, while $\frac{e^x-1}{x}$ does not. This fact implies recursive formulae for B_n.

Lemma 2.2. *For each $m \geqslant 1$, the Bernoulli numbers B_n satisfy the relations*:

(a) $\sum_{n=1}^{m} \binom{m+1}{n} B_n = -1$,

(b) $\sum_{k=1}^{[m/2]}\binom{m+1}{2k}B_{2k}=\frac{m-1}{2}$,
(c) $\sum_{n=1}^{m}(-1)^n\binom{m+1}{n}B_n=m$.

In particular, the first four relations from the item (a) *are*:

$$2B_1=-1,\quad 3B_1+3B_2=-1,\quad 4B_1+6B_2+4B_3=-1,$$
$$5B_1+10B_2+10B_3+5B_4=-1.$$

Proof. (a) One obtains:

$$1=\frac{e^x-1}{x}\cdot\frac{x}{e^x-1}=\left(\sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}\right)\cdot\left(\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n\right)$$
$$=1+\sum_{m=1}^{\infty}\left(\frac{1}{(m+1)!}+\sum_{\binom{k+n=m+1}{k,n\geqslant 1}}\frac{1}{k!}\cdot\frac{B_n}{n!}\right)x^m,$$

i.e.

$$\sum_{n=1}^{m}\frac{1}{(m-n+1)!}\cdot\frac{B_n}{n!}=-\frac{1}{(m+1)!}$$

as required.

(b) Since $B_{2k+1}=0$ for every $k\geqslant 1$, then

$$-1=\sum_{n=1}^{m}\binom{m+1}{n}B_n=(m+1)B_1+\sum_{k=1}^{[m/2]}\binom{m+1}{2k}B_{2k},$$

hence

$$\sum_{k=1}^{[m/2]}\binom{m+1}{2k}B_{2k}=-1+\frac{1}{2}(m+1)=\frac{m-1}{2}.$$

(c) The desired formula is equivalent to the item (b), since $B_{2k+1}=0$ for each $k\geqslant 1$. □

The following theorem is quoted from [21, Corollaries 3.24, 3.25, pp. 77–79].

Theorem 2.3 [1,6,11]. *The Hausdorff series* $H=\log(\exp(P)\cdot\exp(Q))$ *is*

$$H=\sum_{m=0}^{\infty}H_m,$$

where

$$H_0 = Q, \qquad H_1 = P - \frac{1}{2}[Q, P] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\big[Q^{2n}P\big],$$
$$H_m = \frac{1}{m}\left(H_1 \frac{\partial}{\partial Q}\right)(H_{m-1}) \quad \textit{for } m \geqslant 2.$$

2.2. *Extended Bernoulli numbers C_{mn} and compressed variant of CBH formula*

The compressed variant is the case when all commutators commute. To get an explicit form of CBH formula in this setting, one needs to extend the Bernoulli numbers B_n.

Definition 2.4 (extended Bernoulli numbers C_{mn}, generating function $C(\lambda, \mu)$).

(a) Introduce *the extended Bernoulli numbers C_{mn}* in terms of the classical ones:

$$C_{1n} = B_n,$$
$$C_{m+1,n} = \frac{n}{n+1} C_{m,n+1} - \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} B_k C_{m,n-k+1} \quad \text{for } m, n \geqslant 1. \tag{2.4a}$$

The numbers C_{mn} are calculated in Table A.2 of Appendix A for $m + n \leqslant 12$.

(b) Let us introduce *the generating function*

$$C(\lambda, \mu) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{m!n!} \lambda^{n-1} \mu^{m-1}. \tag{2.4b}$$

Formula (2.4a) does not look very naturally. But there is a more natural definition of C_{mn} equivalent to (2.4a), see Proposition 2.8.

Example 2.5. The first few values of the extended Bernoulli numbers are:

$$C_{21} = -\frac{1}{6}, \qquad C_{22} = \frac{1}{6}, \qquad C_{23} = -\frac{1}{15}, \qquad C_{31} = 0, \qquad C_{32} = \frac{1}{15}, \qquad C_{41} = \frac{1}{30}.$$

Then the generating function $C(\lambda, \mu)$ starts with

$$C(\lambda, \mu) = -\frac{1}{2} + \frac{1}{12}(\lambda - \mu) + \frac{1}{24}\lambda\mu + \frac{1}{720}\big(\mu^3 + 4\lambda\mu^2 - 4\lambda^2\mu - \lambda^3\big) + \cdots.$$

Up to degree 10 the function $C(\lambda, \mu)$ is computed in Example A.3.

If W is a word in P, Q, then the expression $[W]$ in a long commutator is regarded as a formal symbol, i.e. $[PQ[W]] := [P, [Q, [W]]]$. But the symbol W in a long commutator is considered as the word in P, Q, for $W = Q^n P^m$ one gets $[WQP] := [Q^n P^m QP]$.

Claim 2.6. *Let P, Q be two elements of a Lie algebra L, W be a word in the letters P, Q.*

(a) *In the quotient $\overline{L} = L/[[L,L],[L,L]]$, for any word W containing at least one letter P and at least one letter Q, one has $[PQ[W]] = [QP[W]]$.*
(b) *In the quotient $\overline{L}$, for any word W containing exactly m letters P and exactly n letters Q, one has $[WQP] = [Q^n P^m QP]$.*

Proof. (a) Since the element $[W]$ contains at least one commutator, then $[[P,Q],[W]] = 0$ in the quotient $\overline{L}$. The Jacobi identity in $\overline{L}$ implies the item (a):

$$\begin{aligned}\big[PQ[W]\big] - \big[QP[W]\big] &= \big[P,\big[Q,[W]\big]\big] + \big[Q,\big[[W],P\big]\big]\\ &= -\big[[W],[P,Q]\big] = 0.\end{aligned}$$

(b) By the item (a) one can permute the letters of W, i.e. one may assume $W = Q^n P^m$. □

Let L be the Lie algebra freely generated by the symbols P, Q. Recall that the series $H_1 \in \widehat{L}$ was introduced in Theorem 2.3. Put $\overline{L} = L/[[L,L],[L,L]]$. As usual by $\widehat{\overline{L}}$ denote the algebra of formal series of elements from $\overline{L}$.

Claim 2.7. *In the algebra $\widehat{\overline{L}}$, for the derivative $D = H_1 \frac{\partial}{\partial Q}$ and all $m, n \geqslant 1$, one has*

$$[H_1, P] = -\sum_{k=1}^{\infty} \frac{B_k}{k!}\big[PQ^kP\big]; \tag{2.7a}$$

$$D\big[Q^nP\big] = (n-1)\big[Q^{n-2}PQP\big] - \sum_{k=1}^{\infty} \frac{B_k}{k!}\big[Q^{n-1}PQ^kP\big]; \tag{2.7b}$$

$$D\big[Q^{n-1}P^{m-1}QP\big] = (n-1)\big[Q^{n-2}P^mQP\big] - \sum_{k=1}^{\infty} \frac{B_k}{k!}\big[Q^{k+n-2}P^mQP\big]. \tag{2.7c}$$

Proof. (a) It suffices to rewrite the formula of H_1 from Theorem 2.3 as follows:

$$\begin{aligned}&H_1 \overset{(2.3)}{=} P + \sum_{k=1}^{\infty} \frac{B_k}{k!}\big[Q^kP\big]\\ &\Rightarrow \quad [H_1, P] = -\left[P, \sum_{k=1}^{\infty} \frac{B_k}{k!}\big[Q^kP\big]\right] = -\sum_{k=1}^{\infty} \frac{B_k}{k!}\big[PQ^kP\big].\end{aligned}$$

Observe that here *the first key point* ($B_n = 0$ for odd $n \geqslant 3$) was used.

(b) Induction on n. The base $n=1$ follows from (a):

$$D[QP] \overset{(2.1c)}{=} [DQ, P] + [Q, DP] = [H_1, P] \overset{(2.7a)}{=} -\sum_{k=1}^{\infty} \frac{B_k}{k!}\big[PQ^kP\big].$$

Induction step (from n to $n+1$):

$$\begin{aligned} D\big[Q^{n+1}P\big] &\overset{(2.1c)}{=} \big[H_1, \big[Q^nP\big]\big] + \big[Q, D\big[Q^nP\big]\big] \\ &= \big[PQ^nP\big] + (n-1)\big[Q^{n-1}PQP\big] - \left[Q, \sum_{k=1}^{\infty} \frac{B_k}{k!}\big[Q^{n-1}PQ^kP\big]\right]. \end{aligned}$$

It remains to apply Claim 2.6(a):

$$\big[PQ^nP\big] + (n-1)\big[Q^{n-1}PQP\big] = n\big[Q^{n-1}PQP\big].$$

The item (b) and

$$D\big[Q^{n-1}P^{m-1}QP\big] \overset{(2.6a)}{=} D\big[P^{m-1}Q^nP\big] \overset{(2.1c)}{=} \underbrace{\big[P, \big[P, \big[\ldots}_{m-1}, D\big[Q^nP\big]\ldots\big]\big]\big]$$

imply (c). □

The following result gives a natural definition of the extended Bernoulli numbers C_{mn}: they give rise to an explicit compressed CBH formula (*the second key point*).

Proposition 2.8 (compressed variant of CBH). *Let L be the Lie algebra freely generated by the symbols P, Q. Under the natural projection $\widehat{L} \to \widehat{\overline{L}}$, where $\overline{L} = L/[[L,L],[L,L]]$, the Hausdorff series $H = \log(\exp(P)\cdot\exp(Q))$ maps onto the series*

$$\overline{H} = P + Q + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{C_{mn}}{m!n!}\big[Q^{n-1}P^{m-1}QP\big]. \tag{2.8}$$

Proof. By Theorem 2.3 the series H maps onto the series $\overline{H} = \sum_{m=0}^{\infty} \overline{H}_m$, where

$$\overline{H}_0 = Q, \qquad \overline{H}_1 = H_1 = P + \sum_{n=1}^{\infty} \frac{B_n}{n!}\big[Q^nP\big],$$

$$\overline{H}_{m+1} = \frac{1}{m!} D^m(H_1) \quad \text{for } m \geqslant 1 \text{ and } D = H_1 \frac{\partial}{\partial Q}.$$

It remains to prove the following formula:

$$D^m(H_1) = \sum_{n=1}^{\infty} \frac{C_{m+1,n}}{n!}\big[Q^{n-1}P^m QP\big] \quad \text{for each } m \geqslant 1. \tag{2.8$'_m$}$$

The base $m = 1$ is completely analogous to the inductive step (from $m-1$ to m):

$$\begin{aligned}
D^m(H_1) &\overset{(2.8'_{m-1})}{=} D\left(\sum_{n=1}^{\infty} \frac{C_{mn}}{n!}\big[Q^{n-1}P^{m-1}QP\big]\right) = \sum_{n=1}^{\infty} \frac{C_{mn}}{n!} D\big[Q^{n-1}P^{m-1}QP\big] \\
&\overset{(2.7c)}{=} \sum_{n=1}^{\infty} \frac{C_{mn}}{n!}\left((n-1)\big[Q^{n-2}P^m QP\big] - \sum_{k=1}^{\infty} \frac{B_k}{k!}\big[Q^{k+n-2}P^m QP\big]\right) \\
&= \sum_{n=0}^{\infty} \frac{C_{m,n+1}}{(n+1)!} n\big[Q^{n-1}P^m QP\big] - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{C_{mn}}{n!}\frac{B_k}{k!}\big[Q^{k+n-2}P^m QP\big] \\
&= \sum_{n=1}^{\infty}\left(\frac{C_{m,n+1}}{(n+1)!} n - \sum_{k=1}^{n} \frac{C_{m,n-k+1}}{(n-k+1)!}\frac{B_k}{k!}\right)\big[Q^{n-1}P^m QP\big] \\
&\overset{(2.4a)}{=} \sum_{n=1}^{\infty} \frac{C_{m+1,n}}{n!}\big[Q^{n-1}P^m QP\big]. \qquad \square
\end{aligned}$$

The series $\overline{H}$ will be calculated up to degree 10 in Proposition A.4.

2.3. Properties of the extended Bernoulli numbers and $C(\lambda, \mu)$

Claim 2.9. *Under the projection $\widehat{L} \to \widehat{\overline{L}}$, where $\overline{L} = L/[[L, L], [L, L]]$, the Hausdorff series $H = \log(\exp(P)\cdot\exp(Q))$ maps onto the series (note that $B_1 = -\frac{1}{2}$, but $C'_{11} = \frac{1}{2}$):*

$$\overline{H} = P + Q + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{C'_{mn}}{m!n!}\big[P^{n-1}Q^{m-1}PQ\big], \quad \textit{where } C'_{11} = \frac{1}{2},\ C'_{1n} = B_n; \tag{2.9a}$$

$$C'_{m+1,n} = \frac{n}{n+1}C'_{m,n+1} - \frac{1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k}C'_{1k}C'_{m,n-k+1} \quad \textit{for } m \geqslant 1,\ n \geqslant 2. \tag{2.9b}$$

Proof. Proof is completely similar to the proof of Proposition 2.8. One can use the following analogue of Theorem 2.3 [21, the remark after Corollary 3.25, p. 80]: the Hausdorff series $H = \log(\exp(P)\cdot\exp(Q))$ is equal to $H = \sum_{m=0}^{\infty} H'_m$, where

$$H'_0 = P, \qquad H'_1 = Q + \frac{1}{2}[P, Q] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\big[P^{2n}Q\big],$$

$$H'_m = \frac{1}{m}\left(H'_1 \frac{\partial}{\partial P}\right)\left(H'_{m-1}\right) \quad \text{for } m \geqslant 2.$$

Here the derivative $D' = H'_1 \frac{\partial}{\partial P}$ maps P onto H'_1 and Q onto 0. To get formula (2.9a) it suffices to apply the equations similar to Claim 2.7, where P, Q are interchanged:

$$\left[H'_1, Q\right] = -\frac{1}{2}[QPQ] - \sum_{k=2}^{\infty} \frac{B_k}{k!}\left[QP^kQ\right] = -\sum_{k=1}^{\infty} \frac{C'_{1k}}{k!}\left[QP^kQ\right]; \qquad (2.7\text{a}')$$

$$D'\left[P^nQ\right] = (n-1)\left[P^{n-2}QPQ\right] - \sum_{k=1}^{\infty} \frac{C'_{1k}}{k!}\left[P^{n-1}QP^kQ\right]; \qquad (2.7\text{b}')$$

$$D'\left[P^{n-1}Q^{m-1}PQ\right] = (n-1)\left[P^{n-2}Q^mPQ\right] - \sum_{k=1}^{\infty} \frac{C'_{1k}}{k!}\left[P^{k+n-2}Q^mPQ\right]. \qquad (2.7\text{c}')$$

Actually, it remains to deduce by $(2.7\text{c}')$ the formula analogous to $(2.8'_m)$:

$$\left(D'\right)^m\left(H'_1\right) = \sum_{n=1}^{\infty} \frac{C'_{m+1,n}}{n!}\left[P^{n-1}Q^mPQ\right] \quad \text{for every } m \geqslant 1. \qquad \square$$

Lemma 2.10. *The extended Bernoulli numbers are symmetric in the following sense*:

$$C_{mn} = (-1)^{m+n} C_{nm} \quad \textit{for all } m, n \geqslant 1. \qquad (2.10)$$

Hence the generating function $C(\lambda, \mu)$ obeys the symmetry $C(\lambda, \mu) = C(-\mu, -\lambda)$.

Proof. Let us rewrite the recursive formula (2.9b) in a more explicit form:

$$C'_{m+1,n} = \frac{n}{n+1} C'_{m,n+1} - \frac{1}{2} C'_{mn} - \frac{1}{n+1} \sum_{k=1}^{[n/2]} \binom{n+1}{2k} B_{2k} C'_{m,n-2k+1}.$$

In the same form formula (2.4a) looks like

$$C_{m+1,n} = \frac{n}{n+1} C_{m,n+1} + \frac{1}{2} C_{mn} - \frac{1}{n+1} \sum_{k=1}^{[n/2]} \binom{n+1}{2k} B_{2k} C_{m,n-2k+1}.$$

If the latter equation is multiplied by $(-1)^{m+n}$, then one obtains

$$(-1)^{m+n} C_{m+1,n} = \frac{n}{n+1}(-1)^{m+n} C_{m,n+1} - \frac{1}{2}(-1)^{m+n-1} C_{mn} - \frac{1}{n+1} \sum_{k=1}^{[n/2]} \binom{n+1}{2k} B_{2k} (-1)^{m+n-2k} C_{m,n-2k+1}.$$

Hence the numbers C'_{mn} and $(-1)^{m+n-1}C_{mn}$ obey the same recursive relation. Since $C'_{1n} = (-1)^n B_n = (-1)^n C_{1n}$ (*the first key point*) for all $n \geqslant 1$, then $C'_{mn} = (-1)^{m+n-1}C_{mn}$ for all $m, n \geqslant 1$. Then formula (2.9a) converts to

$$\begin{aligned} \overline{H} - P - Q &= \sum_{m,n\geqslant 1} (-1)^{m+n-1}\frac{C_{mn}}{m!n!}\left[P^{n-1}Q^{m-1}PQ\right] \\ &\overset{(2.6)}{=} \sum_{m,n\geqslant 1} (-1)^{m+n}\frac{C_{mn}}{m!n!}\left[Q^{m-1}P^{n-1}QP\right]. \end{aligned}$$

The above formula and (2.8) imply $C_{mn} = (-1)^{m+n}C_{nm}$ as required. □

The following two assertions show that the extended Bernoulli numbers C_{mn} are not too complicated. The numbers C_{mn} can be expressed via binoms and the Bernoulli numbers B_n.

Lemma 2.11. *The extended Bernoulli numbers can be expressed via the classical ones*:

$$C_{mn} = \sum_{k=0}^{m-1}\binom{m}{k}B_{n+k} \quad \text{for each } m \geqslant 1. \tag{2.11}$$

In particular, one gets

$$C_{1n} = B_n, \qquad C_{2n} = B_n + 2B_{n+1}, \qquad C_{3n} = B_n + 3B_{n+1} + 3B_{n+2},$$
$$C_{4n} = B_n + 4B_{n+1} + 6B_{n+2} + 4B_{n+3}.$$

Proof. Multiplying (2.4a) by $\frac{n+1}{n}$, one has

$$\frac{n+1}{n}C_{m+1,n} = C_{m,n+1} - \frac{1}{n}\sum_{k=1}^{n}\binom{n+1}{k}B_k C_{m,n-k+1},$$

i.e.

$$C_{m,n+1} = \frac{n+1}{n}C_{m+1,n} + \frac{1}{n}\sum_{k=1}^{n}\binom{n+1}{k}B_k C_{m,n-k+1} \quad \text{for all } m, n \geqslant 1. \tag{2.4a$'$}$$

Equation (2.11) will be checked by induction on m. The base $m = 1$ follows from Definition 2.4(a). Suppose that formula (2.11) holds for m, let us prove it for $m + 1$.

$$\begin{aligned} C_{m+1,n} &\overset{(2.10)}{=} (-1)^{m+n+1}C_{n,m+1} \\ &\overset{(2.4\text{a}')}{=} \frac{(-1)^{m+n+1}}{m}\left((m+1)C_{n+1,m} + \sum_{k=1}^{m}\binom{m+1}{k}B_k C_{n,m-k+1}\right) \end{aligned}$$

$$\overset{(2.10)}{=} (-1)^{m+n+1}\left(\frac{m+1}{m}(-1)^{m+n+1}C_{m,n+1} + \frac{1}{m}\sum_{k=1}^{m}\binom{m+1}{k}B_k(-1)^{m+n-k+1}C_{m-k+1,n}\right)$$

(by hypothesis)

$$= \frac{m+1}{m}\sum_{k=0}^{m-1}\binom{m}{k}B_{n+k+1} + \frac{1}{m}\sum_{k=1}^{m}(-1)^k\binom{m+1}{k}B_k\sum_{l=0}^{m-k}\binom{m-k+1}{l}B_{n+l}$$

$$= \frac{m+1}{m}\sum_{l=1}^{m}\binom{m}{l-1}B_{n+l} + \frac{1}{m}\sum_{k=1}^{m}(-1)^k\binom{m+1}{k}B_kB_n + \frac{1}{m}\sum_{k=1}^{m}(-1)^k\binom{m+1}{k}B_k\sum_{l=1}^{m-k}\binom{m-k+1}{l}B_{n+l}$$

$$= \frac{B_n}{m}\sum_{k=1}^{m}(-1)^k\binom{m+1}{k}B_k + \sum_{l=1}^{m}\frac{B_{n+l}}{m}\left((m+1)\binom{m}{l-1} + \sum_{k=1}^{m-l}(-1)^k\binom{m+1}{k}B_k\binom{m-k+1}{l}\right).$$

Since by Lemma 2.2(c) the first term is equal to B_n, then it remains to check the formula

$$\frac{m+1}{m}\binom{m}{l-1} + \frac{1}{m}\sum_{k=1}^{m-l}(-1)^k\binom{m+1}{k}B_k\binom{m-k+1}{l} = \binom{m+1}{l} \quad \text{for all } m \geqslant l \geqslant 1.$$

The left-hand side of the above equation is

$$\frac{(m+1)\cdot(m-1)!}{(l-1)!(m-l+1)!} + \sum_{k=1}^{m-l}\left((-1)^kB_k\cdot\frac{(m+1)\cdot(m-1)!}{k!(m-k+1)!}\cdot\frac{(m-k+1)!}{l!(m-k-l+1)!}\right)$$

$$= \frac{(m+1)\cdot(m-1)!}{(l-1)!(m-l+1)!} + \frac{(m+1)\cdot(m-1)!}{l!(m-l+1)!}\cdot\sum_{k=1}^{m-l}(-1)^kB_k\binom{m-l+1}{k}$$

$$\overset{(2.2c)}{=} \frac{(m+1)\cdot(m-1)!}{(l-1)!(m-l+1)!}\left(1+\frac{m-l}{l}\right) = \frac{(m+1)!}{l!(m-l+1)!} = \binom{m+1}{l},$$

as required. □

Now one can get an explicit expression of $C(\lambda,\mu)$, which does not follow immediately from Definition 2.4 or Proposition 2.8 (*the third key point*).

Proposition 2.12. *The generating function* $C(\lambda,\mu)$ *from Definition* 2.4(b) *is equal to*

$$C(\lambda,\mu)=\frac{e^{\mu}-1}{\lambda\mu}\cdot\left(\frac{\lambda+\mu}{e^{\lambda+\mu}-1}-\frac{\mu}{e^{\mu}-1}\right). \tag{2.12}$$

Proof. The next trick is applied: to get a non-trivial relation a symmetry is destroyed,

$$\begin{aligned}
\lambda\mu C(\lambda,\mu) &= \sum_{m,n\geqslant 1}\frac{C_{mn}}{m!n!}\lambda^n\mu^m \\
&\overset{(2.11)}{=} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m-1}\binom{m}{k}B_{n+k}\right)\cdot\frac{\lambda^n}{n!}\cdot\frac{\mu^m}{m!} \\
&= \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\sum_{m=k+1}^{\infty}\frac{B_{n+k}\lambda^n\mu^m}{n!(m-k)!k!}=\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{n+k}\frac{\lambda^n\mu^m\mu^k}{n!m!k!} \\
&= \left(\sum_{m=1}^{\infty}\frac{\mu^m}{m!}\right)\left(\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}B_{n+k}\frac{\lambda^n\mu^k}{n!k!}\right) \\
&= \left(e^{\mu}-1\right)\cdot\left(\sum_{n=1}^{\infty}B_n\frac{\lambda^n}{n!}+\sum_{k,n\geqslant 1}B_{n+k}\frac{\lambda^n\mu^k}{n!k!}\right) \\
&\overset{(2.1b)}{=} \left(e^{\mu}-1\right)\cdot\left(\frac{\lambda}{e^{\lambda}-1}-1+\widetilde{C}(\lambda,\mu)\right),
\end{aligned}$$

where

$$\widetilde{C}(\lambda,\mu)=\sum_{k,n\geqslant 1}B_{n+k}\frac{\lambda^n\mu^k}{n!k!}.$$

Since $(-1)^{n+k}B_{n+k}=B_{n+k}$ for $n,k\geqslant 1$ (*the first key point*), then

$$\widetilde{C}(-\mu,-\lambda)=\sum_{k,n\geqslant 1}(-1)^{n+k}B_{n+k}\frac{\lambda^k\mu^n}{k!n!}=\sum_{k,n\geqslant 1}B_{n+k}\frac{\lambda^k\mu^n}{k!n!}=\widetilde{C}(\lambda,\mu).$$

One obtains

$$\begin{aligned}
\left(e^{\mu}-1\right)\cdot\left(\frac{\lambda}{e^{\lambda}-1}-1+\widetilde{C}(\lambda,\mu)\right) &= \lambda\mu C(\lambda,\mu)\overset{(2.10)}{=}\lambda\mu C(-\mu,-\lambda) \\
&= \left(e^{-\lambda}-1\right)\cdot\left(\frac{-\mu}{e^{-\mu}-1}-1+\widetilde{C}(-\mu,-\lambda)\right) \\
&= \left(e^{-\lambda}-1\right)\cdot\left(\frac{-\mu}{e^{-\mu}-1}-1+\widetilde{C}(\lambda,\mu)\right),
\end{aligned}$$

hence

$$\begin{aligned}\left(e^{-\lambda}-e^{\mu}\right)\widetilde{C}(\lambda,\mu)&=\left(e^{\mu}-1\right)\left(\frac{\lambda}{e^{\lambda}-1}-1\right)-\left(e^{-\lambda}-1\right)\left(\frac{-\mu}{e^{-\mu}-1}-1\right)\\&=\lambda\frac{e^{\mu}-1}{e^{\lambda}-1}+\mu\frac{e^{-\lambda}-1}{e^{-\mu}-1}+e^{-\lambda}-e^{\mu}.\end{aligned}$$

By substituting the above formula into the expression of $\lambda\mu C(\lambda,\mu)$ via $\widetilde{C}(\lambda,\mu)$, one gets

$$C(\lambda,\mu)=\frac{e^{\mu}-1}{\lambda\mu}\cdot\left(\frac{\lambda}{e^{\lambda}-1}+\left(\lambda\frac{e^{\mu}-1}{e^{\lambda}-1}+\mu\frac{e^{-\lambda}-1}{e^{-\mu}-1}\right):\left(e^{-\lambda}-e^{\mu}\right)\right).$$

All quotients are well defined as formal Laurent series. It remains to prove:

$$\lambda\frac{e^{\mu}-1}{e^{\lambda}-1}+\mu\frac{e^{-\lambda}-1}{e^{-\mu}-1}=\left(e^{-\lambda}-e^{\mu}\right)\cdot\left(\frac{\lambda+\mu}{e^{\lambda+\mu}-1}-\frac{\lambda}{e^{\lambda}-1}-\frac{\mu}{e^{\mu}-1}\right),\quad\text{or}$$

$$\lambda e^{\lambda}\frac{e^{\mu}-1}{e^{\lambda}-1}+\mu e^{\mu}\frac{1-e^{\lambda}}{1-e^{\mu}}=\left(1-e^{\lambda+\mu}\right)\cdot\left(\frac{\lambda+\mu}{e^{\lambda+\mu}-1}-\frac{\lambda}{e^{\lambda}-1}-\frac{\mu}{e^{\mu}-1}\right),\quad\text{or}$$

$$\begin{aligned}&\left(e^{\mu}-1\right)\frac{\lambda e^{\lambda}}{e^{\lambda}-1}+\left(e^{\lambda}-1\right)\frac{\mu e^{\mu}}{e^{\mu}-1}\\&\quad=-\lambda-\mu-\left(1-e^{\lambda+\mu}\right)\cdot\left(\frac{\lambda}{e^{\lambda}-1}+\frac{\mu}{e^{\mu}-1}\right),\quad\text{or}\end{aligned}$$

$$\begin{aligned}&\left(e^{\mu}-1\right)\cdot\left(\frac{\lambda}{e^{\lambda}-1}+\lambda\right)+\left(e^{\lambda}-1\right)\cdot\left(\frac{\mu}{e^{\mu}-1}+\mu\right)\\&\quad=-\lambda-\mu+\left(e^{\lambda+\mu}-1\right)\cdot\left(\frac{\lambda}{e^{\lambda}-1}+\frac{\mu}{e^{\mu}-1}\right),\quad\text{or}\end{aligned}$$

$$\lambda e^{\mu}+\mu e^{\lambda}=\left(e^{\lambda+\mu}-e^{\mu}\right)\frac{\lambda}{e^{\lambda}-1}+\left(e^{\lambda+\mu}-e^{\lambda}\right)\frac{\mu}{e^{\mu}-1},$$

which is clear. □

3. Compressed hexagon equation

In this section one shall write explicitly the compressed hexagon and prove Theorem 1.5(a).

3.1. *First simplification of the hexagon*

Lemma 3.1 gives a first simplification of the hexagon (1.3b). Due to this *4th key point* it remains to apply CBH formula exactly once.

Lemma 3.1. *In $\widehat{A}_3$ the hexagon equation* (1.3b) *is equivalent to the following equation*:

$$\exp(a+b+c) = \exp\bigl(\psi(c,a)\bigr)\cdot\exp\bigl(\psi(b,c)\bigr)\cdot\exp\bigl(\psi(a,b)\bigr), \tag{3.1}$$

where $\psi(a,b) := \log(\exp(\varphi(a,b))\cdot\exp(a)) \in \widehat{L}_3$, $a := t^{12}$, $b := t^{23}$, $c := t^{13}$.

Proof. Let us rewrite the original hexagon (1.3b) in a more explicit form:

$$\exp\bigl(t^{13}+t^{23}\bigr) = \exp\bigl(\varphi\bigl(t^{13},t^{12}\bigr)\bigr)\cdot\exp\bigl(t^{13}\bigr)\cdot\exp\bigl(-\varphi\bigl(t^{13},t^{23}\bigr)\bigr)$$
$$\times\exp\bigl(t^{23}\bigr)\cdot\exp\bigl(\varphi\bigl(t^{12},t^{23}\bigr)\bigr).$$

Introduce the symbols $a = t^{12}$, $b = t^{23}$, $c = t^{13}$ and apply the symmetry $\varphi(b,c) = -\varphi(c,b)$.

$$\exp(b+c) = \exp\bigl(\varphi(c,a)\bigr)\cdot\exp(c)\cdot\exp\bigl(\varphi(b,c)\bigr)\cdot\exp(b)\cdot\exp\bigl(\varphi(a,b)\bigr).$$

It remains to multiply the above equation by $\exp(a)$ from the right. The element a commutes with $b+c$ in L_3 by definition. Moreover, $a+b+c$ is a central element of L_3. □

Recall, $\widehat{\overline{L}}_3$ is the algebra of formal series of elements from $\overline{L}_3 = L_3/[[L_3,L_3],[L_3,L_3]]$.

Claim 3.2. *In $\widehat{\overline{L}}_3$, for the compressed series $\bar\psi(a,b) = \log(\exp(\bar\varphi(a,b))\cdot\exp(a))$, one has*

$$\bar\psi(a,b) = a + \sum_{n=0}^{\infty}\frac{B_n}{n!}\bigl[a^n\bar\varphi(a,b)\bigr]$$
$$= a + \bar\varphi(a,b) - \frac{1}{2}\bigl[a,\bar\varphi(a,b)\bigr] + \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}\bigl[a^{2n}\bar\varphi(a,b)\bigr]. \tag{3.2}$$

Proof. Apply Proposition 2.8 for the Hausdorff series $\bar\psi(a,b) = \log(\exp(\bar\varphi(a,b))\cdot\exp(a))$ and $Q = a$, $P = \bar\varphi(a,b)$. Since all commutators included in $\bar\varphi(a,b)$ commute with each other in $\overline{L}_3$, then $[Q^nP] = [a^n\bar\varphi(a,b)]$ and $[Q^{n-1}P^{m-1}QP] = 0$ for all $m \geqslant 2$, $n \geqslant 1$. □

In particular, by Claim 3.2 the series $\bar\psi(a,b)$ starts with

$$\bar\psi(a,b) = a + \bar\varphi(a,b) - \frac{1}{2}\bigl[a,\bar\varphi(a,b)\bigr] + \frac{1}{12}\bigl[a,\bigl[a,\bar\varphi(a,b)\bigr]\bigr]$$
$$-\frac{1}{720}\bigl[a,\bigl[a,\bigl[a,\bigl[a,\varphi(a,b)\bigr]\bigr]\bigr]\bigr] + \cdots.$$

3.2. The Lie algebra $L_3(\lambda, \mu)$

Recall that $\overline{L}_3 = L_3/[[L_3, L_3], [L_3, L_3]]$ by definition.

Definition 3.3 (Lie algebra $L_3(\lambda, \mu)$). Let us introduce the linear space $L_3(\lambda, \mu)$ generated by a, b, and $\lambda^k\mu^l[ab]$ for $k, l \geqslant 0$, where λ, μ are formal parameters. At this moment the expression $[ab]$ means a formal symbol, which denotes an element of $L_3(\lambda, \mu)$. Define in $L_3(\lambda, \mu)$ *the bracket* by $[a, b] := [ab]$, $[a, \lambda^k\mu^l[ab]] := \lambda^{k+1}\mu^l[ab]$, $[b, \lambda^k\mu^l[ab]] := \lambda^k\mu^{l+1}[ab]$, the other brackets are zero. One can check that this bracket satisfies the Jacobi identity. Actually, only the following identities (up to permutation $a \leftrightarrow b$) contain non-zero terms:

$$\big[a, \big[a, \lambda^k\mu^l[ab]\big]\big] + \big[a, \big[\lambda^k\mu^l[ab], a\big]\big] = 0,$$

$$\big[a, \big[b, \lambda^k\mu^l[ab]\big]\big] + \big[b, \big[\lambda^k\mu^l[ab], a\big]\big] + \big[\lambda^k\mu^l[ab], [ab]\big] = 0.$$

Hence the space $L_3(\lambda, \mu)$ becomes a Lie algebra. Now the formal symbol $[ab]$ is the truly commutator $[a, b] \in L_3(\lambda, \mu)$. Note that in $L_3(\lambda, \mu)$ one has $[a^kb^lab] = \lambda^k\mu^l[a, b]$.

By $\widehat{L}_3(\lambda, \mu)$ denote the algebra of formal series of elements from $L_3(\lambda, \mu)$.

Recall the notation $[a_1a_2 \dots a_k] := [a_1, [a_2, [\dots, a_k] \dots]]$ from Definition 1.2(c).

Proposition 3.4.

(a) *Let $L(a + b + c) \subset L_3$ be the* 1*-dimensional Lie subalgebra generated by the element $a + b + c$, $L(a, b) \subset L_3$ be the Lie subalgebra freely generated by a, b. Then the Lie algebra L_3 is isomorphic to the direct sum $L(a + b + c) \oplus L(a, b)$.*
(b) *The quotient $\overline{L}(a, b) := L(a, b)/[[L(a, b), L(a, b)], [L(a, b), L(a, b)]]$ is isomorphic to the Lie algebra $L_3(\lambda, \mu)$ introduced in Definition* 3.3*.*
(c) *The quotient $\overline{L}_3 = L_3/[[L_3, L_3], [L_3, L_3]]$ is isomorphic to $L(a + b + c) \oplus L_3(\lambda, \mu)$.*

Proof. (a) In the setting $a := t^{12}$, $b := t^{23}$, $c := t^{13}$, $a + b + c$ is a central element of L_3. Moreover, the defining relations of the Lie algebra L_3 become $[a, a + b + c] = [b, a + b + c] = 0$. Now the isomorphism $L_3 \cong L(a + b + c) \oplus L(a, b)$ is obvious.

(b) By definition the Lie algebra $L(a, b)$ is linearly generated by a, b and all commutators $[wab]$, where w is a word in a, b. By Claim 2.6(b) the quotient $\overline{L}(a, b)$ is linearly generated by a, b and commutators $[a^kb^lab]$ with $k, l \geqslant 0$. The only non-zero brackets of these elements in $\overline{L}(a, b)$ are $[a, b]$, $[a, a^kb^lab] = [a^{k+1}b^lab]$, $[b, a^kb^lab] = [a^kb^{l+1}ab]$. One has $[[L_3(\lambda, \mu), L_3(\lambda, \mu)], [L_3(\lambda, \mu), L_3(\lambda, \mu)]] = 0$. The required isomorphism $\overline{L}(a, b) \to L_3(\lambda, \mu)$ is defined by $[a^kb^lab] \mapsto \lambda^k\mu^l[a, b]$.

The item (c) follows from (a) and (b). □

Observe that in $\widehat{L}_3(\lambda, \mu)$ any series of commutators $[a^kb^lab]$ is (a series in λ, μ)$\times[a, b]$. It turns out that Theorem 1.5(a) follows from conditions (1.3a), (1.3d), (1.3e).

Proof of Theorem 1.5(a). By condition (1.3d) a Drinfeld associator $\Phi(a,b)$ (hence also $\varphi(a,b)=\log\Phi(a,b)$ and $\bar\varphi(a,b)$) does not contain terms with a^k, b^l. Then by Proposition 3.4(c) a compressed associator $\bar\varphi(a,b)$ consists of commutators, i.e.

$$\bar\varphi(a,b)=\sum_{k,l\geqslant 0}\alpha_{kl}\big[a^k b^l ab\big],\quad \alpha_{kl}\in\mathbb{C}.$$

The symmetry (1.3a) implies $\bar\varphi(a,b)=-\bar\varphi(b,a)$, therefore $\alpha_{kl}=\alpha_{lk}$ for all $k,l\geqslant 0$. □

Definition 3.5 (Generating series $f(\lambda,\mu)$ and $g(\lambda,\mu)$). Let $\mathbb{C}[\![\lambda,\mu]\!]$ be the set of formal power series with complex coefficients in the commuting arguments λ,μ. Introduce the series $f,g\in\mathbb{C}[\![\lambda,\mu]\!]$ by the formulae in the algebra $\widehat{L}_3(\lambda,\mu)$

$$\bar\varphi(a,b)=\sum_{k,l\geqslant 0}\alpha_{kl}\big[a^k b^l ab\big]=f(\lambda,\mu)\cdot[a,b],\qquad \bar\psi(a,b)=a+g(\lambda,\mu)\cdot[a,b],$$

respectively. Theorem 1.5(a) secures that there is a desired symmetric series $f(\lambda,\mu)=f(\mu,\lambda)$. A series $g(\lambda,\mu)$ exists due to Claim 3.2.

Claim 3.6.

(a) *The series $f(\lambda,\mu)$ and $g(\lambda,\mu)$ are related as follows*:

$$\begin{aligned} g(\lambda,\mu) &= \sum_{k=0}^{\infty}\frac{B_k}{k!}\lambda^k\cdot f(\lambda,\mu)=\left(1-\frac{\lambda}{2}+\frac{\lambda^2}{12}-\frac{\lambda^4}{720}+\cdots\right)f(\lambda,\mu)\\ &\overset{(2.1b)}{=}\frac{\lambda f(\lambda,\mu)}{e^\lambda-1}. \end{aligned}\tag{3.6a}$$

(b) *In the algebra $\widehat{L}_3(\lambda,\mu)$ one has*

$$\begin{aligned}\bar\psi(b,c)&=b+g(\mu,\rho)\cdot[a,b],\\ \bar\psi(c,a)&=c+g(\rho,\lambda)\cdot[a,b],\quad \textit{where } \rho=-\lambda-\mu.\end{aligned}\tag{3.6b}$$

Proof. The following calculations imply the item (a):

$$\begin{aligned} g(\lambda,\mu)[ab] &\overset{(3.5)}{=}\bar\psi(a,b)-a\overset{(3.2)}{=}\sum_{k=0}\frac{B_k}{k!}\big[a^k\bar\varphi(a,b)\big]\\ &\overset{(3.5)}{=}\sum_{k=0}\frac{B_k}{k!}\big[a^k f(\lambda,\mu)[ab]\big]\overset{(3.3)}{=}\sum_{k=0}\frac{B_k}{k!}\lambda^k f(\lambda,\mu)\cdot[ab]. \end{aligned}$$

The item (b) follows from Definition 3.5 and

$$\bar\varphi(b,c)=f(\mu,\rho)[ab],\qquad \bar\varphi(c,a)=f(\rho,\lambda)[ab].$$

The latter formula is proved similarly by using (1.2a) $[ab] = [bc] = [ca]$, namely:

$$\bar{\varphi}(b,c) \overset{(3.5)}{=} \sum_{k,l\geqslant 0} \alpha_{kl}\big[b^k c^l bc\big] \overset{(1.2a)}{=} \sum_{k,l\geqslant 0} \alpha_{kl}\big[b^k(-a-b)^l ab\big]$$
$$\overset{(3.3)}{=} \sum_{k,l\geqslant 0} \alpha_{kl}\mu^k(-\lambda-\mu)^l[ab] \overset{(3.5)}{=} f(\mu,\rho)[ab]. \qquad \square$$

Example 3.7. For the series $\bar{\varphi}^{\mathrm{B}}(a,b) = f^{\mathrm{B}}(\lambda,\mu)\cdot[a,b]$ from Example 1.4, one has:

$$\alpha_{00} = \frac{1}{6}; \qquad \alpha_{10} = \alpha_{01} = 0; \qquad \alpha_{20} = \alpha_{02} = -\frac{1}{90},$$
$$\alpha_{11} = -\frac{1}{360}; \qquad \alpha_{30} = \alpha_{21} = \alpha_{12} = \alpha_{03} = 0.$$

Then up to degree 3 the corresponding series $f^{\mathrm{B}}(\lambda,\mu)$ and $g^{\mathrm{B}}(\lambda,\mu)$ are:

$$f^{\mathrm{B}}(\lambda,\mu) = \frac{1}{6} - \frac{4\lambda^2+\lambda\mu+4\mu^2}{360},$$
$$g^{\mathrm{B}}(\lambda,\mu) = \frac{1}{6} - \frac{\lambda}{12} + \frac{\lambda^2-\lambda\mu-4\mu^2}{360} + \frac{4\lambda^3+\lambda^2\mu+4\lambda\mu^2}{720}.$$

Moreover, one can compute the series from Claim 3.6(b) (they will be needed later):

$$g^{\mathrm{B}}(\mu,\rho) = \frac{1}{6} - \frac{\mu}{12} - \frac{4\lambda^2+7\lambda\mu+2\mu^2}{360} + \frac{4\lambda^2\mu+7\lambda\mu^2+7\mu^3}{720} + \cdots,$$
$$g^{\mathrm{B}}(\rho,\lambda) = \frac{1}{6} + \frac{\lambda+\mu}{12} - \frac{2\lambda^2+3\lambda\mu+\mu^2}{360} - \frac{7\lambda^3+14\lambda^2\mu+11\lambda\mu^2+4\mu^3}{720} + \cdots,$$

hence

$$G^{\mathrm{B}}(\lambda,\mu) := g^{\mathrm{B}}(\lambda,\mu) + g^{\mathrm{B}}(\mu,\rho) + g^{\mathrm{B}}(\rho,\lambda)$$
$$= \frac{1}{2} - \frac{\lambda^2+\lambda\mu+\mu^2}{72} + \frac{\mu^3-3\lambda^2\mu-\lambda^3}{240} + \cdots,$$
$$T^{\mathrm{B}}(\lambda,\mu) := 1 + \lambda g^{\mathrm{B}}(\mu,\rho) - \mu g^{\mathrm{B}}(\lambda,\mu)$$
$$= 1 + \frac{\lambda-\mu}{6} + \frac{4\mu^3-\lambda\mu^2-8\lambda^2\mu-4\lambda^3}{360} + \cdots.$$

3.3. *Explicit form of the compressed hexagon* $\overline{(1.3\mathrm{b})}$

The compressed hexagon can be rewritten as an equation in the algebra $\widehat{L}_3(\lambda,\mu)$. This simplification (*the 5th key point*) allows to solve completely the compressed hexagon.

Claim 3.8. *For the generators $a = t^{12}$, $b = t^{23}$, $c = t^{13}$ of the Lie algebra L_3, set $P := \bar{\psi}(b, c)$, $Q := \bar{\psi}(a, b)$. Then in the algebra $\widehat{L}_3(\lambda, \mu)$ one has (see $g(\lambda, \mu)$ in Definition* 3.5):

$$[Q, P] = T(\lambda, \mu) \cdot [a, b], \quad \text{where } T(\lambda, \mu) = 1 + \lambda g(\mu, -\lambda - \mu) - \mu g(\lambda, \mu). \tag{3.8}$$

Proof. By Definition 3.4(a) and Claim 3.6 one gets

$$\begin{aligned}[Q, P] &= \big[a + g(\lambda, \mu)[ab], b + g(\mu, -\lambda - \mu)[ab]\big] \\ &= [ab] + g(\mu, -\lambda - \mu)[aab] - g(\lambda, \mu)[bab].\end{aligned}$$

It remains to use the relations $[aab] = \lambda[ab]$ and $[bab] = \mu[ab]$ of $L_3(\lambda, \mu)$. □

Proposition 3.9. *Let $\bar{\varphi} = \sum_{k,l \geqslant 0} \alpha_{kl}[a^k b^l ab]$ be a compressed Drinfeld associator, $\alpha_{kl} \in \mathbb{C}$,*

$$f(\lambda, \mu) = \sum_{k,l \geqslant 0} \alpha_{kl} \lambda^k \mu^l$$

be the generating function of the coefficients α_{kl}. Then the compressed hexagon $\overline{(1.3b)}$ is equivalent to the following equation in the algebra $\mathbb{C}[\![\lambda, \mu]\!]$:

$$G(\lambda, \mu) + C(\lambda, \mu) \cdot T(\lambda, \mu) = 0,$$

where

$$\begin{aligned} G(\lambda, \mu) &:= g(\lambda, \mu) + g(\mu, \rho) + g(\rho, \lambda), \\ T(\lambda, \mu) &:= 1 + \lambda g(\mu, \rho) - \mu g(\lambda, \mu), \quad \rho := -\lambda - \mu. \end{aligned} \tag{3.9}$$

Proof. Let us apply Proposition 2.8 for the Hausdorff series $\overline{H} = \log(\exp(P) \cdot \exp(Q))$, where $P = \bar{\psi}(b, c) = b + g(\mu, \rho)[a, b]$, $Q = \bar{\psi}(a, b) = a + g(\lambda, \mu)[a, b]$. One has

$$\overline{H} = P + Q + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{mn}}{m!n!} \big[Q^{n-1} P^{m-1} Q P\big].$$

Since in the quotient $\overline{L}_3$ all commutators commute, then formula (3.8) implies

$$\big[Q^{n-1} P^{m-1} Q P\big] = T(\lambda, \mu) \cdot \big[a^{n-1} b^{m-1} ab\big] = \lambda^{n-1} \mu^{m-1} T(\lambda, \mu) \cdot [a, b],$$

hence

$$\overline{H} = a + g(\lambda,\mu)[a,b] + b + g(\mu,\rho)[a,b] + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{C_{mn}}{m!n!}\lambda^{n-1}\mu^{m-1}T(\lambda,\mu)\cdot[a,b]$$

(by Definition 2.4(b))

$$= a + b + \big(g(\lambda,\mu) + g(\mu,\rho)\big)\cdot[a,b] + C(\lambda,\mu)T(\lambda,\mu)\cdot[a,b].$$

On the other hand, by Lemma 3.1 one has $\overline{H} = \log(\exp(-\bar{\psi}(c,a))\cdot\exp(a+b+c))$. But $a+b+c$ is a central element of L_3, hence $\overline{H} = a+b+c-\bar{\psi}(c,a)$.

Let us take together both above expressions for $\overline{H}$ and use Claim 3.6(b) for $\bar{\psi}(c,a)$:

$$\begin{aligned} &a + b + \big(g(\lambda,\mu) + g(\mu,\rho)\big)[ab] + C(\lambda,\mu)T(\lambda,\mu)[ab] \\ &\quad = a + b + c - \big(c + g(\rho,\lambda)[ab]\big) \\ &\quad \Leftrightarrow \quad (3.9). \qquad \square \end{aligned}$$

Example 3.10. By $G_k^{\mathrm{B}}, C_k, T_k^{\mathrm{B}}$ denote the degree k parts of the functions $G^{\mathrm{B}}(\lambda,\mu)$, $C(\lambda,\mu)$, $T^{\mathrm{B}}(\lambda,\mu)$, respectively. Due to Examples 2.5 and 3.7 one can calculate

$$G_0^{\mathrm{B}}(\lambda,\mu) = \frac{1}{2}, \qquad G_1^{\mathrm{B}}(\lambda,\mu) = 0, \qquad G_2^{\mathrm{B}}(\lambda,\mu) = -\frac{\lambda^2+\lambda\mu+\mu^2}{72},$$

$$G_3^{\mathrm{B}}(\lambda,\mu) = \frac{\mu^3 - 3\lambda^2\mu - \lambda^3}{240};$$

$$C_0(\lambda,\mu) = -\frac{1}{2}, \qquad C_1(\lambda,\mu) = \frac{\lambda-\mu}{12}, \qquad C_2(\lambda,\mu) = \frac{\lambda\mu}{24},$$

$$C_3(\lambda,\mu) = \frac{\mu^3 + 4\lambda\mu^2 - 4\lambda^2\mu - \lambda^3}{720};$$

$$T_0^{\mathrm{B}}(\lambda,\mu) = 1, \qquad T_1^{\mathrm{B}}(\lambda,\mu) = \frac{\lambda-\mu}{6}, \qquad T_2^{\mathrm{B}}(\lambda,\mu) = 0,$$

$$T_3^{\mathrm{B}}(\lambda,\mu) = \frac{4\mu^3 - \lambda\mu^2 - 8\lambda^2\mu - 4\lambda^3}{360}.$$

Now one can check by hands the first four compressed hexagons:

$$G_0^{\mathrm{B}} + C_0 = 0, \qquad G_1^{\mathrm{B}} + C_0T_1^{\mathrm{B}} + C_1 = 0, \qquad G_2^{\mathrm{B}} + C_1T_1^{\mathrm{B}} + C_2 = 0,$$

$$G_3^{\mathrm{B}} + C_0T_3^{\mathrm{B}} + C_2T_1^{\mathrm{B}} + C_3 = 0.$$

4. Solving the compressed hexagon equation

Here one solves completely the compressed hexagon $\overline{(1.3\mathrm{b})} = (3.9)$ from Proposition 3.9.

4.1. Further simplifications of the compressed hexagon

Lemma 4.1. *Let*

$$f(\lambda,\mu)=\sum_{k,l\geqslant 0}\alpha_{kl}\lambda^k\mu^l$$

be the generating function of the coefficients α_{kl} *of a compressed associator* $\bar{\varphi}\in\widehat{\overline{L}}_3$. *Then the compressed hexagon* $\overline{(1.3\mathrm{b})}=(3.9)$ *is equivalent to Eq.* (4.1a) *and to Eq.* (4.1b) *in the algebra* $\mathbb{C}[\![\lambda,\mu]\!]$:

$$\frac{\lambda f(\lambda,\mu)}{e^\lambda-1}\big(1-\mu C(\lambda,\mu)\big)+\frac{\mu f(\mu,\rho)}{e^\mu-1}\big(1+\lambda C(\lambda,\mu)\big)+\frac{\rho f(\rho,\lambda)}{e^\rho-1}+C(\lambda,\mu)=0,\quad (4.1\mathrm{a})$$

where

$$\rho=-\lambda-\mu,\qquad C(\lambda,\mu)=\frac{e^\mu-1}{\lambda\mu}\cdot\left(\frac{\lambda+\mu}{e^{\lambda+\mu}-1}-\frac{\mu}{e^\mu-1}\right);$$

$$f(\lambda,\mu)+e^\mu f(\mu,-\lambda-\mu)+e^{-\lambda}f(\lambda,-\lambda-\mu)=\frac{1}{\lambda+\mu}\left(\frac{e^\mu-1}{\mu}+\frac{e^{-\lambda}-1}{\lambda}\right).\quad (4.1\mathrm{b})$$

Proof. (a) Let us rewrite Eq. (3.9) as follows:

$$g(\lambda,\mu)\cdot\big(1-\mu C(\lambda,\mu)\big)+g(\mu,\rho)\cdot\big(1+\lambda C(\lambda,\mu)\big)+g(\rho,\lambda)+C(\lambda,\mu)=0.$$

The above formula and Claim 3.6(a) imply Eq. (4.1a).

(b) By using the formula (2.12) in the form

$$C(\lambda,\mu)=\frac{e^\mu-1}{\lambda\mu}\cdot\frac{\lambda+\mu}{e^{\lambda+\mu}-1}-\frac{1}{\lambda},$$

one gets

$$\frac{\mu}{e^\mu-1}\big(1+\lambda C(\lambda,\mu)\big)=\frac{\mu}{e^\mu-1}\left(\frac{e^\mu-1}{\mu}\cdot\frac{\lambda+\mu}{e^{\lambda+\mu}-1}\right)=\frac{\lambda+\mu}{e^{\lambda+\mu}-1}.$$

Now apply the symmetry (*the third key point*)

$$C(\lambda,\mu)=C(-\mu,-\lambda)=\frac{e^{-\lambda}-1}{\lambda\mu}\cdot\frac{-\lambda-\mu}{e^{-\lambda-\mu}-1}+\frac{1}{\mu},$$

hence

$$\frac{\lambda}{e^\lambda - 1}\big(1-\mu C(\lambda,\mu)\big) = \frac{\lambda}{e^\lambda-1}\left(\frac{e^{-\lambda}-1}{\lambda}\cdot\frac{\lambda+\mu}{e^{-\lambda-\mu}-1}\right)$$
$$= \frac{1-e^\lambda}{e^\lambda-1}\cdot e^\mu\frac{\lambda+\mu}{1-e^{\lambda+\mu}} = e^\mu\frac{\lambda+\mu}{e^{\lambda+\mu}-1}.$$

Then Eq. (4.1a) converts to

$$e^\mu\frac{\lambda+\mu}{e^{\lambda+\mu}-1}f(\lambda,\mu) + \frac{\lambda+\mu}{e^{\lambda+\mu}-1}f(\mu,\rho) - \frac{\lambda+\mu}{e^{-\lambda-\mu}-1}f(\rho,\lambda)$$
$$+\frac{e^\mu-1}{\lambda\mu}\cdot\frac{\lambda+\mu}{e^{\lambda+\mu}-1} - \frac{1}{\lambda} = 0, \quad \text{or}$$
$$e^\mu\frac{\lambda+\mu}{e^{\lambda+\mu}-1}f(\lambda,\mu) + \frac{\lambda+\mu}{e^{\lambda+\mu}-1}f(\mu,\rho) + e^{\lambda+\mu}\frac{\lambda+\mu}{e^{\lambda+\mu}-1}f(\rho,\lambda)$$
$$= \frac{1}{\lambda}\left(1 - \frac{e^\mu-1}{\mu}\cdot\frac{\lambda+\mu}{e^{\lambda+\mu}-1}\right).$$

One can multiply both sides by $\frac{e^{\lambda+\mu}-1}{\lambda+\mu}$ and obtain

$$e^\mu f(\lambda,\mu) + f(\mu,\rho) + e^{\lambda+\mu}f(\rho,\lambda) = \frac{1}{\lambda}\left(\frac{e^{\lambda+\mu}-1}{\lambda+\mu} - \frac{e^\mu-1}{\mu}\right).$$

Let us swap λ and $\rho = -\lambda-\mu$ (in other words one substitutes $(-\lambda-\mu)$ for λ):

$$e^\mu f(-\lambda-\mu,\mu) + f(\mu,\lambda) + e^{-\lambda}f(\lambda,-\lambda-\mu) = \frac{1}{\lambda+\mu}\left(\frac{e^{-\lambda}-1}{\lambda} + \frac{e^\mu-1}{\mu}\right).$$

To get Eq. (4.1b) it remains to use the symmetry $f(\lambda,\mu) = f(\mu,\lambda)$. □

Proof of Theorem 1.5(b). It follows from Lemma 4.1(b): the compressed hexagon equation $\overline{(1.3b)}$ is equivalent to (4.1b) = (1.5b). □

An explicit form of the compressed hexagon $\overline{(1.3b)}$ is *the* 6*th key point*.

Lemma 4.2. *For the generating function $f(\lambda,\mu)$ of any compressed associator, one has*

$$Even\big(f(\lambda,0)\big) = \frac{1}{2\lambda^2}\left(\frac{2\lambda}{e^{2\lambda}-1} + \lambda - 1\right),$$
$$f(\lambda,-\lambda) = \frac{1}{\lambda^2} - \frac{2}{\lambda(e^\lambda - e^{-\lambda})}. \tag{4.2}$$

In particular, one obtains the extreme coefficients $\alpha_{2k,0} = \frac{2^{2k+1}B_{2k+2}}{(2k+2)!}$ for every $k \geqslant 0$.

Proof. Let us solve Eq. (4.1b) explicitly for $\mu = 0$ (then set $\frac{e^{\mu}-1}{\mu}|_{\mu=0} = 1$). One has

$$f(\lambda,0) + f(0,-\lambda) + e^{-\lambda} f(\lambda,-\lambda) = \frac{1}{\lambda}\left(1 + \frac{e^{-\lambda}-1}{\lambda}\right) = \frac{e^{-\lambda}+\lambda-1}{\lambda^2}.$$

Since

$$Odd\big(f(\lambda,-\lambda)\big) = 0, \qquad Odd\big(f(\lambda,0) + f(0,-\lambda)\big) = 0, \quad \text{and}$$
$$Even\big(f(0,-\lambda)\big) = Even\big(f(\lambda,0)\big) = \sum_{k=0}^{\infty} \alpha_{2k,0}\lambda^{2k},$$

then one obtains

$$2\sum_{k=0}^{\infty} \alpha_{2k,0}\lambda^{2k} + e^{-\lambda} Even\big(f(\lambda,-\lambda)\big) = \frac{e^{-\lambda}+\lambda-1}{\lambda^2}.$$

By substituting $(-\lambda)$ for λ and using $Even(f(-\lambda,\lambda)) = Even(f(\lambda,-\lambda))$, one gets

$$2\sum_{k=0}^{\infty} \alpha_{2k,0}\lambda^{2k} + e^{\lambda} Even\big(f(\lambda,-\lambda)\big) = \frac{e^{\lambda}-\lambda-1}{\lambda^2}.$$

From the two above equations one deduces

$$f(\lambda,-\lambda) = Even\big(f(\lambda,-\lambda)\big) = \frac{1}{\lambda^2} - \frac{2}{\lambda(e^{\lambda}-e^{-\lambda})} \quad \text{and}$$
$$Even\big(f(\lambda,0)\big) = \sum_{k=0}^{\infty} \alpha_{2k,0}\lambda^{2k} = \frac{1}{2\lambda^2}\left(\frac{2\lambda}{e^{2\lambda}-1} + \lambda - 1\right) = \frac{1}{2\lambda^2}\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(2\lambda)^{2n}. \qquad \square$$

Example 4.3. Note that

$$\alpha_{2k,0} = \frac{2^{2k+1}B_{2k+2}}{(2k+2)!}$$

are correct coefficients of the logarithm $\varphi(a,b)$ of any honest Drinfeld associator $\Phi(a,b)$ from $\widehat{A}_3$, not only the compressed one.

This form of the extreme coefficients $\alpha_{2k,0}$ is *the* 7*th key point*:

$$\alpha_{00} = \frac{2B_2}{2!} = \frac{1}{6}, \qquad \alpha_{20} = \frac{2^3 B_4}{4!} = -\frac{1}{90}, \qquad \alpha_{40} = \frac{2^5 B_6}{6!} = \frac{1}{945},$$
$$\alpha_{60} = \frac{2^7 B_8}{8!} = -\frac{1}{9450}.$$

For the coefficients γ_k of the series

$$\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}=\sum_{k=0}^{\infty}\gamma_k\lambda^{2k},$$

similarly to Lemma 2.2 one can find the recursive formula

$$\sum_{k=0}^{n}\frac{\gamma_{n-k}}{(2k+1)!}=0\quad\text{for each } n>1,\ \gamma_0=0.$$

Hence

$$\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}=1-\frac{1}{6}\lambda^2+\frac{7}{360}\lambda^4-\frac{31}{3\cdot 7!}\lambda^6+\frac{127}{15\cdot 8!}\lambda^8+\cdots. \tag{4.3}$$

For the generating function of any compressed Drinfeld associator $\bar{\varphi}\in\widehat{\overline{L}}_3$, one gets

$$f(\lambda,-\lambda)=\frac{1}{6}-\frac{7}{360}\lambda^2+\frac{31}{15120}\lambda^4-\frac{127}{604800}\lambda^6+\cdots.$$

Lemma 4.4. *Under the conditions of Lemma* 4.1, *set* $\tilde{f}(\lambda,\mu):=1+\lambda\mu f(\lambda,\mu)$.

(a) *The new function* $\tilde{f}(\lambda,\mu)$ *obeys the same symmetry*: $\tilde{f}(\lambda,\mu)=\tilde{f}(\mu,\lambda)$. *Moreover, the function* $f(\lambda,\mu)$ *is even* (*i.e.* $f(\lambda,\mu)=f(-\lambda,-\mu)$) *if and only if* $\tilde{f}(\lambda,\mu)$ *is even.*
(b) *The compressed hexagon* $\overline{(1.3\mathrm{b})}=(4.1\mathrm{b})$ *is equivalent to the following equation in* $\mathbb{C}[\![\lambda,\mu]\!]$:

$$(\lambda+\mu)\tilde{f}(\lambda,\mu)=\lambda e^{\mu}\tilde{f}(\mu,-\lambda-\mu)+\mu e^{-\lambda}\tilde{f}(\lambda,-\lambda-\mu). \tag{4.4}$$

Proof. The item (a) follows directly from the definition $\tilde{f}(\lambda,\mu)=1+\lambda\mu f(\lambda,\mu)$.
(b) Equation (4.1b) can be rewritten as follows:

$$\left(f(\lambda,\mu)+\frac{1}{\lambda\mu}\right)+e^{\mu}\left(f(\mu,-\lambda-\mu)-\frac{1}{\mu(\lambda+\mu)}\right)$$
$$+e^{-\lambda}\left(f(\lambda,-\lambda-\mu)-\frac{1}{\lambda(\lambda+\mu)}\right)=0.$$

To get (4.4) it remains to multiply the above equation by $\lambda\mu(\lambda+\mu)$. □

Recall that the even and odd parts of a series were introduced before Theorem 1.5.

Lemma 4.5. *In the notations of Lemma* 4.4, *the compressed hexagon* $\overline{(1.3\mathrm{b})}=(4.4)$ *splits into the two following equations in the algebra* $\mathbb{C}[\![\lambda,\mu]\!]$:

$$(\lambda+\mu)Even\big(\tilde f(\lambda,\mu)\big)=\lambda e^{\mu}Even\big(\tilde f(\mu,-\lambda-\mu)\big)+\mu e^{-\lambda}Even\big(\tilde f(\lambda,-\lambda-\mu)\big);\tag{4.5a}$$

$$Odd\big(f(\lambda,\mu)\big)+e^{\mu}Odd\big(f(\mu,-\lambda-\mu)\big)+e^{-\lambda}Odd\big(f(\lambda,-\lambda-\mu)\big)=0.\tag{4.5b}$$

Proof. Let us substitute $(-\mu,-\lambda)$ for (λ,μ) into (4.4) and use $f(-\mu,-\lambda)=f(-\lambda,-\mu)$

$$-(\lambda+\mu)\tilde f(-\lambda,-\mu)=-\mu e^{-\lambda}\tilde f(-\lambda,\lambda+\mu)-\lambda e^{\mu}\tilde f(-\mu,\lambda+\mu).$$

If one subtracts the latter equation from (4.4), then after dividing by 2 one obtains Eq. (4.5a). If one adds the latter equation to (4.4), then one has

$$(\lambda+\mu)Odd\big(\tilde f(\lambda,\mu)\big)=\lambda e^{\mu}Odd\big(\tilde f(\mu,-\lambda-\mu)\big)+\mu e^{-\lambda}Odd\big(\tilde f(\lambda,-\lambda-\mu)\big).$$

Since $Odd(\tilde f(\lambda,\mu))=\lambda\mu Odd(f(\lambda,\mu))$, after dividing by $\lambda\mu(\lambda+\mu)$ one gets (4.5b). □

The splitting of Lemma 4.5 is *the 8th key point*.

4.2. Explicit description of all even compressed associators

Lemma 4.6. *The general solution of Eq.* (4.5a) *is*

$$Even\big(\tilde f(\lambda,\mu)\big)=1+\lambda\mu\cdot Even\big(f(\lambda,\mu)\big)=\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}h(\lambda,\mu),\tag{4.6}$$

where $h(\lambda,\mu)$ is a function satisfying the boundary condition $h(\lambda,0)=\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}$ and the symmetry relations $h(\lambda,\mu)=h(\mu,\lambda)=h(-\lambda,-\mu)=h(\lambda,-\lambda-\mu)$.

Proof. Let us swap the arguments λ,μ in (4.5a). Due to $\tilde f(\lambda,\mu)=\tilde f(\mu,\lambda)$ one gets

$$(\lambda+\mu)Even\big(\tilde f(\lambda,\mu)\big)=\mu e^{\lambda}Even\big(\tilde f(\lambda,-\lambda-\mu)\big)+\lambda e^{-\mu}Even\big(\tilde f(\mu,-\lambda-\mu)\big).$$

By subtracting the above equation from (4.5a) one obtains

$$\lambda\big(e^{\mu}-e^{-\mu}\big)\cdot Even\big(\tilde f(\mu,-\lambda-\mu)\big)+\mu\big(e^{-\lambda}-e^{\lambda}\big)\cdot Even\big(\tilde f(\lambda,-\lambda-\mu)\big)=0,\quad\text{or}$$

$$Even\big(\tilde f(\mu,-\lambda-\mu)\big):\left(\frac{e^{\lambda}-e^{-\lambda}}{2\lambda}\right)=Even\big(\tilde f(\lambda,-\lambda-\mu)\big):\left(\frac{e^{\mu}-e^{-\mu}}{2\mu}\right).$$

Introduce the function

$$\begin{aligned}h(\lambda,\mu)&:=Even\big(\tilde f(\mu,-\lambda-\mu)\big):\left(\frac{e^{\lambda}-e^{-\lambda}}{2\lambda}\right)\\&=Even\big(\tilde f(\lambda,-\lambda-\mu)\big):\left(\frac{e^{\mu}-e^{-\mu}}{2\mu}\right).\end{aligned}\tag{4.6$'$}$$

Then the new function is even and symmetric: $h(\lambda,\mu)=h(\mu,\lambda)=h(-\lambda,-\mu)$. Let us substitute the expressions

$$Even\big(\tilde f(\mu,-\lambda-\mu)\big)=\left(\frac{e^{\lambda}-e^{-\lambda}}{2\lambda}\right)\cdot h(\lambda,\mu),$$
$$Even\big(\tilde f(\lambda,-\lambda-\mu)\big)=\left(\frac{e^{\mu}-e^{-\mu}}{2\mu}\right)\cdot h(\lambda,\mu)$$

into Eq. (4.5a). One has

$$(\lambda+\mu)Even\big(\tilde f(\lambda,\mu)\big)=\left(\lambda e^{\mu}\frac{e^{\lambda}-e^{-\lambda}}{2\lambda}+\mu e^{-\lambda}\frac{e^{\mu}-e^{-\mu}}{2\mu}\right)h(\lambda,\mu)$$
$$=\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2}h(\lambda,\mu).$$

So, (4.6) is proved. Let us substitute $(-\lambda-\mu)$ for μ into the above equation:

$$Even\big(\tilde f(\lambda,-\lambda-\mu)\big)=\frac{e^{-\mu}-e^{\mu}}{2(-\mu)}h(\lambda,-\lambda-\mu)=\frac{e^{\mu}-e^{-\mu}}{2\mu}h(\lambda,-\lambda-\mu).$$

The last formula and Eq. (4.6′) imply $h(\lambda,\mu)=h(\lambda,-\lambda-\mu)$. To obtain the condition $h(\lambda,0)=\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}$ it remains to substitute $\mu=0$ into (4.6). Finally, by using $h(\lambda,\mu)=h(\lambda,-\lambda-\mu)=h(\mu,-\lambda-\mu)$ it is easy to check that (4.6) gives a solution of (4.5a)

$$(\lambda+\mu)\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}h(\lambda,\mu)=\lambda e^{\mu}\frac{e^{-\lambda}-e^{\lambda}}{2(-\lambda)}h(\mu,-\lambda-\mu)$$
$$+\mu e^{-\lambda}\frac{e^{-\mu}-e^{\mu}}{2(-\mu)}h(\lambda,-\lambda-\mu).\qquad\square$$

Observe that the conditions of Lemma 4.6 imply also $h(\lambda,-\lambda)=h(\lambda,0)=\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}$. This agrees with the formula (4.2):

$$h(\lambda,-\lambda)\overset{(4.6)}{=}1-\lambda^2 f(\lambda,-\lambda)\overset{(4.2)}{=}\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}.$$

So, the compressed hexagon $\overline{(1.3b)}$ was reduced to $h(\lambda,\mu)=h(\lambda,-\lambda-\mu)$, this is *the 9th key point*. To describe all series $h(\lambda,\mu)$ with this symmetry one needs to introduce associator polynomials.

Definition 4.7 (Associator polynomials). For each $n\geqslant 0$, a homogeneous polynomial

$$F_n(\lambda,\mu)=\sum_{k=0}^{n}\delta_k\lambda^k\mu^{n-k}\quad\text{with } \delta_k\in\mathbb{C}$$

is called *an associator polynomial,* if the relations $F_n(\lambda,\mu)=F_n(\mu,\lambda)=F_n(\lambda,-\lambda-\mu)$ hold for all $\lambda,\mu\in\mathbb{C}$.

The Drinfeld series $s(\lambda,\mu)$ from Definition 6.1(b) contains *the Drinfeld associator polynomials* $F^{\mathrm{D}}_{2n+1}(\lambda,\mu)=(\lambda+\mu)^{2n+1}-\lambda^{2n+1}-\mu^{2n+1}$. They are used in the proof of Corollary 1.6(c).

Example 4.8. In degrees 0, 2, 4 an associator polynomial is unique up to a factor:

$$F_0(\lambda,\mu)=1,\qquad F_2(\lambda,\mu)=\lambda^2+\lambda\mu+\mu^2,$$
$$F_4(\lambda,\mu)=\lambda^4+2\lambda^3\mu+3\lambda^2\mu^2+2\lambda\mu^3+\mu^4.$$

In each odd degree $n=1,3,5$ there is also exactly one associator polynomial:

$$F_1(\lambda,\mu)=0,\qquad F_3(\lambda,\mu)=\lambda^2\mu+\lambda\mu^2,\qquad F_5(\lambda,\mu)=\lambda^4\mu+2\lambda^3\mu^2+2\lambda^2\mu^3+\lambda\mu^5.$$

Rather surprisingly, that an associator polynomial of the degree n is not unique in general, for instance, in degree 6. One can check by hands that in degree 6 there is a 1-parametric family of associator polynomials:

$$F_6(\lambda,\mu)=\lambda^6+3\lambda^5\mu+\delta\lambda^4\mu^2+(2\delta-5)\lambda^3\mu^3+\delta\lambda^2\mu^4+3\lambda\mu^5+\mu^6,\quad \delta\in\mathbb{C}.$$

However, in degree 7 there is a unique associator polynomial accurate to a factor:

$$F_7(\lambda,\mu)=\lambda^6\mu+3\lambda^5\mu^2+5\lambda^4\mu^3+5\lambda^3\mu^4+3\lambda^2\mu^5+\lambda\mu^6.$$

Lemma 4.9. *Any series $\bar h(\lambda,\mu)$ satisfying the boundary conditions*

$$\bar h(\lambda,0)=\sum_{n=0}^{\infty}\gamma_n\lambda^{2n}$$

and the symmetry relations $\bar h(\lambda,\mu)=\bar h(\mu,\lambda)=\bar h(\lambda,-\lambda-\mu)$ is

$$\bar h(\lambda,\mu)=\sum_{n=0}^{\infty}\gamma_n F_n(\lambda,\mu),$$

where $F_n(\lambda,\mu)$ is an associator polynomial with the extreme coefficient 1 (*respectively* 0) *for each even* (*respectively odd*) $n\geqslant 0$.

Proof. It follows from Definition 4.7. The relation $F_{2n+1}(\lambda,\mu)=F_{2n+1}(\lambda,-\lambda-\mu)$ implies that the extreme coefficient of any associator polynomial $F_{2n+1}(\lambda,\mu)$ is always 0. □

The non-uniqueness of even compressed associators will follow from the non-uniqueness of associator polynomials (*the 10th key point*), see the hexagon part of Theorem 1.5(c) at the end of Section 4.3.

Proposition 4.10.

(a) *For each* $n \geqslant 0$*, any associator polynomial of the degree* $2n$ *is*

$$F_{2n}(\lambda,\mu)=\sum_{k=0}^{[n/3]}\beta_{nk}\lambda^{2k}\mu^{2k}(\lambda+\mu)^{2k}\big(\lambda^2+\lambda\mu+\mu^2\big)^{n-3k}, \tag{4.10a}$$

where $\beta_{nk}\in\mathbb{C}$ *are free parameters for* $0\leqslant k\leqslant[\frac{n}{3}]$.

(b) *For every* $n\geqslant 1$*, any associator polynomial* $F_{2n+1}(\lambda,\mu)$ *has the following form*:

$$F_{2n+1}(\lambda,\mu)=\sum_{k=0}^{[(n-1)/3]}\tilde\beta_{nk}\lambda^{2k+1}\mu^{2k+1}(\lambda+\mu)^{2k+1}\big(\lambda^2+\lambda\mu+\mu^2\big)^{n-3k-1}, \tag{4.10b}$$

where $\tilde\beta_{nk}\in\mathbb{C}$ *are free parameters for* $0\leqslant k\leqslant[\frac{n-1}{3}]$.

Proof. (a) Induction on n. The bases $n=0,1,2$ are in Example 4.8.

Induction step goes down from n to $n-3$.

Suppose that $F_{2n}(\lambda,\mu)$ is an associator polynomial of the degree $2n$ with the extreme coefficient β_{n0}. Then the polynomial $\widetilde F_{2n}(\lambda,\mu)=F_{2n}(\lambda,\mu)-\beta_{n0}(\lambda^2+\lambda\mu+\mu^2)^n$ satisfies Definition 4.7. The relations $F_{2n}(\lambda,\mu)=F_{2n}(\mu,\lambda)=F_{2n}(\lambda,-\lambda-\mu)$ imply

$$F_{2n}(\lambda,\mu)=\beta_{n0}\lambda^{2n}+n\beta_{n0}\lambda^{2n-1}\mu+\cdots+n\beta_{n0}\lambda\mu^{2n-1}+\beta_{n0}\mu^{2n}.$$

The polynomial $\beta_{n0}(\lambda^2+\lambda\mu+\mu^2)^n$ has the same form. Hence the first two (and the last two) coefficients of $\widetilde F_{2n}(\lambda,\mu)$ are always zero.

One gets $\widetilde F_{2n}(\lambda,\mu)=\lambda^2\mu^2\overline F_{2n-4}(\lambda,\mu)$ for a polynomial $\overline F_{2n-4}(\lambda,\mu)$ of the degree $2n-4$, such that $\overline F_{2n-4}(\lambda,\mu)=\overline F_{2n-4}(\mu,\lambda)$ and $\lambda^2\mu^2\overline F_{2n-4}(\lambda,\mu)=\lambda^2(-\lambda-\mu)^2\times \overline F_{2n-4}(\lambda,-\lambda-\mu)$.

The equation $\mu^2\overline F_{2n-4}(\lambda,\mu)=(\lambda+\mu)^2\overline F_{2n-4}(\lambda,-\lambda-\mu)$ implies that there is a polynomial $\overline{\overline F}_{2n-6}(\lambda,\mu)$ of the degree $2n-6$, such that $\overline F_{2n-4}(\lambda,\mu)=(\lambda+\mu)^2\overline{\overline F}_{2n-6}(\lambda,\mu)$.

Moreover, the new polynomial $\overline{\overline F}_{2n-6}(\lambda,\mu)$ satisfies all the conditions of Definition 4.7, i.e. $\overline{\overline F}_{2n-6}(\lambda,\mu)$ is equal to an associator polynomial $F_{2n-6}(\lambda,\mu)$. Hence

$$F_{2n}(\lambda,\mu)-\beta_{n0}\big(\lambda^2+\lambda\mu+\lambda\mu\big)^n=\lambda^2\mu^2(\lambda+\mu)^2F_{2n-6}(\lambda,\mu).$$

The induction hypothesis

$$F_{2n-6}(\lambda,\mu)=\sum_{k=0}^{[n/3]-1}\beta_{n-3,k}\lambda^{2k}\mu^{2k}(\lambda+\mu)^{2k}\big(\lambda^2+\lambda\mu+\mu^2\big)^{n-3k-3}$$

implies

$$F_{2n}(\lambda,\mu)=\beta_{n0}\big(\lambda^2+\lambda\mu+\mu^2\big)^n + \lambda^2\mu^2(\lambda+\mu)^2\sum_{k=0}^{[n/3]-1}\beta_{n-3,k}\lambda^{2k}\mu^{2k}(\lambda+\mu)^{2k}\big(\lambda^2+\lambda\mu+\mu^2\big)^{n-3-3k}.$$

To get (4.10a) it remains to set

$$\beta_{n,k+1}:=\beta_{n-3,k}\quad \text{for } 0\leqslant k\leqslant\left[\frac{n}{3}\right]-1.$$

(b) The proof is analogous to the item (a). If $F_{2n+1}(\lambda,\mu)$ is an associator polynomial, then its extreme coefficient is zero, i.e. $F_{2n+1}(\lambda,\mu)=\lambda\mu\overline{F_{2n-1}}(\lambda,\mu)$ for some polynomial $\overline{F}_{2n-1}(\lambda,\mu)$ of the degree $2n-1$ with the properties

$$\overline{F}_{2n-1}(\lambda,\mu)=\overline{F}_{2n-1}(\mu,\lambda)\quad\text{and}\quad \lambda\mu\overline{F}_{2n-1}(\lambda,\mu)=\lambda(-\lambda-\mu)\overline{F}_{2n-1}(\lambda,-\lambda-\mu).$$

The equation $\mu\overline{F}_{2n-1}(\lambda,\mu)=-(\lambda+\mu)\overline{F}_{2n-1}(\lambda,-\lambda-\mu)$ implies that there is a polynomial $\overline{\overline{F}}_{2n-2}(\lambda,\mu)$ of the degree $2n-2$, such that $\overline{F}_{2n-1}(\lambda,\mu)=(\lambda+\mu)\overline{\overline{F}}_{2n-2}(\lambda,\mu)$. Moreover, the new polynomial $\overline{\overline{F}}_{2n-2}(\lambda,\mu)$ satisfies all the conditions of Definition 4.7, i.e. $\overline{\overline{F}}_{2n-2}(\lambda,\mu)$ is equal to an associator polynomial $F_{2n-2}(\lambda,\mu)$.

To get (4.10b) it remains to apply the formula (4.10a) for the polynomial $F_{2n-2}(\lambda,\mu)$ and to set $\tilde{\beta}_{nk}:=\beta_{n-1,k}$ for $0\leqslant k\leqslant[\frac{n-1}{3}]$. □

The 11th key point: the symmetry $F_n(\lambda,\mu)=F_n(\lambda,-\lambda-\mu)$ has led to (4.10a) and (4.10b).

Example 4.11. By Proposition 4.10 the number of free parameters, on which the family $F_{2n}(\lambda,\mu)$ depends, increases by 1 when n increases by 3. The first six (starting with 0) associator polynomials are unique up to a factor, but not the seventh $F_6(\lambda,\mu)$. One gets:

$$\begin{aligned}
F_6(\lambda,\mu)&=\beta_{30}\big(\lambda^2+\lambda\mu+\mu^2\big)^3+\beta_{31}\lambda^2\mu^2(\lambda+\mu)^2,\\
F_8(\lambda,\mu)&=\beta_{40}\big(\lambda^2+\lambda\mu+\mu^2\big)^4+\beta_{41}\lambda^2\mu^2(\lambda+\mu)^2\big(\lambda^2+\lambda\mu+\mu^2\big),\\
F_{10}(\lambda,\mu)&=\beta_{50}\big(\lambda^2+\lambda\mu+\mu^2\big)^5+\beta_{51}\lambda^2\mu^2(\lambda+\mu)^2\big(\lambda^2+\lambda\mu+\mu^2\big)^2,\\
F_{12}(\lambda,\mu)&=\beta_{60}\big(\lambda^2+\lambda\mu+\mu^2\big)^6+\beta_{61}\lambda^2\mu^2(\lambda+\mu)^2\big(\lambda^2+\lambda\mu+\mu^2\big)^3\\
&\quad+\beta_{62}\lambda^4\mu^4(\lambda+\mu)^4;\\
F_7(\lambda,\mu)&=\tilde{\beta}_{30}\lambda\mu(\lambda+\mu)\big(\lambda^2+\lambda\mu+\mu^2\big)^2,\\
F_9(\lambda,\mu)&=\tilde{\beta}_{40}\lambda\mu(\lambda+\mu)\big(\lambda^2+\lambda\mu+\mu^2\big)^3+\tilde{\beta}_{41}\lambda^3\mu^3(\lambda+\mu)^3,\\
F_{11}(\lambda,\mu)&=\tilde{\beta}_{50}\lambda\mu(\lambda+\mu)\big(\lambda^2+\lambda\mu+\mu^2\big)^4+\tilde{\beta}_{51}\lambda^3\mu^3(\lambda+\mu)^3\big(\lambda^2+\lambda\mu+\mu^2\big),\\
F_{13}(\lambda,\mu)&=\tilde{\beta}_{60}\lambda\mu(\lambda+\mu)\big(\lambda^2+\lambda\mu+\mu^2\big)^5+\tilde{\beta}_{61}\lambda^3\mu^3(\lambda+\mu)^3\big(\lambda^2+\lambda\mu+\mu^2\big)^2.
\end{aligned}$$

One can check that the parameter β_{31} is related with δ from Example 4.8 as $\delta = \beta_{31} + 6$. Up to degree 8 the function $h(\lambda, \mu)$ from Lemma 4.6 is

$$h(\lambda,\mu) = 1 - \frac{1}{6}F_2(\lambda,\mu) + \frac{7}{360}F_4(\lambda,\mu) - \frac{31}{3\cdot 7!}F_6(\lambda,\mu) + \frac{127}{15\cdot 8!}F_8(\lambda,\mu).$$

4.3. Description of the odd parts of all compressed associators

Lemma 4.12. *The general solution of Eq.* (4.5b) *is*

$$Odd\big(f(\lambda,\mu)\big) = \frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2}\tilde{h}(\lambda,\mu), \tag{4.12}$$

where $\tilde{h}(\lambda,\mu)$ is a function satisfying the relations $\tilde{h}(\lambda,\mu) = \tilde{h}(\mu,\lambda) = \tilde{h}(-\lambda,-\mu) = \tilde{h}(\lambda,-\lambda-\mu)$.

Proof. It is analogous to the proof of Lemma 4.6. Let us swap the arguments λ, μ in Eq. (4.5b). Due to the symmetry $f(\lambda,\mu) = f(\mu,\lambda)$ one gets

$$Odd\big(f(\lambda,\mu)\big) + e^{\lambda} Odd\big(f(\lambda,-\lambda-\mu)\big) + e^{-\mu} Odd\big(f(\mu,-\lambda-\mu)\big) = 0.$$

By subtracting the above equation from (4.5b) one obtains

$$\big(e^{\mu} - e^{-\mu}\big)\cdot Odd\big(f(\mu,-\lambda-\mu)\big) + \big(e^{-\lambda} - e^{\lambda}\big)\cdot Odd\big(f(\lambda,-\lambda-\mu)\big) = 0, \quad \text{or}$$

$$Odd\big(f(\mu,-\lambda-\mu)\big) : \left(\frac{e^{\lambda} - e^{-\lambda}}{2}\right) = Odd\big(f(\lambda,-\lambda-\mu)\big) : \left(\frac{e^{\mu} - e^{-\mu}}{2}\right).$$

Introduce the function

$$\begin{aligned}\tilde{h}(\lambda,\mu) &:= -Odd\big(f(\mu,-\lambda-\mu)\big) : \left(\frac{e^{\lambda} - e^{-\lambda}}{2}\right)\\ &= -Odd\big(f(\lambda,-\lambda-\mu)\big) : \left(\frac{e^{\mu} - e^{-\mu}}{2}\right).\end{aligned} \tag{4.12′}$$

Then the new function is even and symmetric: $\tilde{h}(\lambda,\mu) = \tilde{h}(\mu,\lambda) = \tilde{h}(-\lambda,-\mu)$. Let us substitute the expressions

$$Odd\big(f(\mu,-\lambda-\mu)\big) = -\left(\frac{e^{\lambda} - e^{-\lambda}}{2}\right)\cdot\tilde{h}(\lambda,\mu),$$

$$Odd\big(f(\lambda,-\lambda-\mu)\big) = -\left(\frac{e^{\mu} - e^{-\mu}}{2}\right)\cdot\tilde{h}(\lambda,\mu)$$

into Eq. (4.5b). One has

$$Odd\big(f(\lambda,\mu)\big) = \left(e^{\mu}\frac{e^{\lambda}-e^{-\lambda}}{2} + e^{-\lambda}\frac{e^{\mu}-e^{-\mu}}{2}\right)\tilde{h}(\lambda,\mu) = \frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2}\tilde{h}(\lambda,\mu).$$

So, (4.12) is proved. Let us substitute $(-\lambda-\mu)$ for μ into the above equation:

$$Odd\big(f(\lambda,-\lambda-\mu)\big) = -\frac{e^{\mu}-e^{-\mu}}{2}\tilde{h}(\lambda,-\lambda-\mu).$$

The last formula and (4.12′) imply $\tilde{h}(\lambda,\mu)=\tilde{h}(\lambda,-\lambda,-\mu)$. Finally, by using $\tilde{h}(\lambda,\mu)=\tilde{h}(\lambda,-\lambda-\mu)=\tilde{h}(\mu,-\lambda-\mu)$ it is easy to verify that (4.12) gives a solution of (4.5b):

$$\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2}\tilde{h}(\lambda,\mu) + e^{\mu}\frac{e^{-\lambda}-e^{\lambda}}{2}\tilde{h}(\mu,-\lambda-\mu) + e^{-\lambda}\frac{e^{-\mu}-e^{\mu}}{2}\tilde{h}(\lambda,-\lambda-\mu) = 0. \qquad \square$$

The hexagon part of Theorem 1.5(c). By Lemmas 4.6, 4.9, 4.12, and Proposition 4.10(a), the general solution of the equation (1.5b) = (4.1b) is $f(\lambda,\mu) = Even(f(\lambda,\mu)) + Odd(f(\lambda,\mu))$, where

$$1+\lambda\mu\cdot Even\big(f(\lambda,\mu)\big) = \frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}\beta_{nk}\lambda^{2k}\mu^{2k}(\lambda+\mu)^{2k}\omega^{2n-6k}\right),$$

$$Odd\big(f(\lambda,\mu)\big) = \frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}\tilde{\beta}_{nk}\lambda^{2k}\mu^{2k}(\lambda+\mu)^{2k}\omega^{2n-6k}\right),$$

$$\omega = \sqrt{\lambda^2+\lambda\mu+\mu^2}.$$

Let us substitute $\mu=-\lambda$ (then $\omega=\lambda$) into the former equation and apply Lemma 4.2:

$$\sum_{n=0}^{\infty}\beta_{n0}\lambda^{2n} = 1-\lambda^2 Even\big(f(\lambda,-\lambda)\big) \overset{(4.2)}{=} \frac{2\lambda}{e^{\lambda}-e^{-\lambda}}, \quad \text{hence}$$

$$\sum_{n=0}^{\infty}\beta_{n0}\omega^{2n} = \frac{2\omega}{e^{\omega}-e^{-\omega}}.$$

So, in the formula for $Even(f(\lambda,\mu))$ the terms with $k=0$ are replaced by $\frac{2\omega}{e^{\omega}-e^{-\omega}}$. $\square$

5. Compressed pentagon equation

This section is devoted to the proof of the pentagon part of Theorem 1.5(c). It turns out that if all commutators commute, then the compressed pentagon $\overline{(1.3c)}$ follows from the symmetry $\alpha_{kl}=\alpha_{lk}$ (*the 12th key point*), see Proposition 5.10.

5.1. Generators and relations of the quotient $\overline{L}_4$

Here one studies the quotient $\overline{L}_4$, where the compressed pentagon $\overline{(1.3c)} = (5.9)$ lives.

Definition 5.1 (alphabet $\mathbb{L}$, algebras L_4 and $\overline{L}_4$, simple and non-simple commutators).

(a) Let the Lie algebra L_4 be generated by the letters of the alphabet

$$\mathbb{L} = \{a := t^{12},\ b := t^{23},\ c := t^{13},\ d := t^{24},\ e := t^{34},\ v := t^{14}\}$$

and the relations

$$[a,e] = [b,v] = [c,d] = 0 \quad \text{and}$$
$$\begin{cases} x := [a,b] = [b,c] = [c,a], & y := [a,d] = [d,v] = [v,a], \\ z := [b,e] = [e,d] = [d,b], & u := [c,e] = [e,v] = [v,c]. \end{cases}$$

The Lie algebra L_4 is graded by $\deg(s) = 1$ for any letter $s \in \mathbb{L}$. By $\widehat{\overline{L}}_4$ denote the algebra of formal series of elements from $\overline{L}_4 = L_4/[[L_4, L_4], [L_4, L_4]]$.

(b) Let w be a word in the alphabet $\mathbb{L} = \{t^{ij},\ 1 \leqslant i < j \leqslant 4\}$. Let I_w be the set formed by the upper indices of the letters t^{ij} included in w. If I_w contains at most three different indices, then the commutator $[w] \in \overline{L}_4$ is called *simple*, otherwise $[w]$ is *non-simple*.

For example, the commutators $[aab]$ and $[vad]$ are simple, but $[dab]$ is not simple.

Claim 5.2. *The following relations hold in the quotient* $\overline{L}_4$:

(a) $[a+b+c,x] = 0$, $[a+d+v,y] = 0$, $[b+e+d,z] = 0$, $[c+e+v,u] = 0$;
(b) $[da]+[ea]+[va] = 0$, $[db]+[eb]+[vb] = 0$, $[dx]+[ex]+[vx] = 0$;
(c) $[dx] = -[cy] = -[cz] = [du]$, $[ex] = -[ey] = -[az] = [au]$, $[vx] = -[by] = -[vz] = [bu]$.

Proof. (a) By Definition 5.1(a) one gets $[a+b+c,a] = [c,a]-[a,b] \overset{(5.1a)}{=} 0$ and similarly $[a+b+c,b] \overset{(5.1a)}{=} 0$, hence $[a+b+c,x] = 0$. The other relations are proved analogously.

(b) By Definition 5.1(a) one has $[d+e+v,a] = (-[a,d]+[v,a]) - [a,e] \overset{(5.1a)}{=} 0$ and similarly $[d+e+v,b] \overset{(5.1a)}{=} 0$, hence $[d+e+v,x] = 0$.

(c) Definition 5.1(a), the Jacobi identity (J), and $[cd] = 0$ are used below:

$$[dx] \overset{(5.1a)}{=} [dbc] \overset{(J)}{=} -[bcd] - [cdb] \overset{(5.1a)}{=} -[cz],$$
$$[dx] \overset{(5.1a)}{=} [dab] \overset{(J)}{=} -[abd] - [bda] \overset{(5.1a)}{=} [az] + [by],$$
$$[dx] \overset{(5.1a)}{=} [dca] \overset{(J)}{=} -[adc] - [cad] \overset{(5.1a)}{=} -[cy],$$
$$[cy] \overset{(5.1a)}{=} [cva] \overset{(J)}{=} -[vac] - [acv] \overset{(5.1a)}{=} [vx] + [au],$$

$$[cy] \overset{(5.1a)}{=} [cdv] \overset{(J)}{=} -[dvc] - [vcd] \overset{(5.1a)}{=} -[du],$$
$$[cz] \overset{(5.1a)}{=} [cbe] \overset{(J)}{=} -[bec] - [ecb] \overset{(5.1a)}{=} [bu] + [ex].$$

One gets

$$[dx] = -[cy] = -[cz] = [du] = [az] + [by] = -[bu] - [ex] = -[vx] - [au],$$

i.e.

$$[bu] = -[dx] - [ex] \overset{(5.2b)}{=} [vx] \quad \text{and} \quad [au] = -[dx] - [vx] \overset{(5.2b)}{=} [ex].$$

Similarly, by $[ae] = 0$ one has:

$$[ex] \overset{(5.1a)}{=} [eab] \overset{(J)}{=} -[abe] - [bea] \overset{(5.1a)}{=} -[az],$$
$$[az] \overset{(5.1a)}{=} [aed] \overset{(J)}{=} -[eda] - [dae] \overset{(5.1a)}{=} [ey].$$

Then

$$[ex] = -[az] = -[ey] \quad \text{and} \quad [by] = [dx] - [az] = [dx] + [ex] = -[vx].$$

Finally, one has

$$[du] \overset{(5.1a)}{=} [dev] \overset{(J)}{=} -[vde] - [evd] \overset{(5.1a)}{=} [vz] + [ey]$$
$$\Rightarrow \quad [vz] = [du] - [ey] = [dx] + [ex] = -[vx]. \qquad \square$$

Claim 5.3.

(a) *Every simple (respectively non-simple) commutator* $[w] \in \overline{L}_4$ *of degree* 3 *can be expressed linearly via* $[ax]$, $[bx]$, $[ay]$, $[dy]$, $[bz]$, $[ez]$, $[cu]$, $[eu]$ (*respectively via* $[dx]$ *and* $[ex]$).

(b) *The degree* 3 *part of* $\overline{L}_4$ *is linearly generated by* 8 *simple commutators* $[ax]$, $[bx]$, $[ay]$, $[dy]$, $[bz]$, $[ez]$, $[cu]$, $[eu]$ *and* 2 *non-simple ones* $[dx]$, $[ex]$.

Proof. (a) The degree 3 part of $\overline{L}_4$ contains exactly:

- 12 simple commutators: $[ax]$, $[bx]$, $[cx]$, $[ay]$, $[dy]$, $[vy]$, $[bz]$, $[ez]$, $[dz]$, $[cu]$, $[eu]$, $[vu]$; and
- 12 non-simple ones: $[dx]$, $[ex]$, $[vx]$, $[by]$, $[cy]$, $[ey]$, $[az]$, $[cz]$, $[vz]$, $[au]$, $[bu]$, $[du]$.

Due to the relations (5.2a) one can throw out the four simple commutators $[cx]$, $[vy]$, $[dz]$, and $[vu]$. By the relations (5.2c) any non-simple commutator reduces to $[dx]$, $[ex]$, and $[vx]$. To eliminate $[vx]$ it remains to apply the relation (5.2b): $[vx] = -[dx] - [ex]$.

(b) Any element of $\overline{L}_4$ is a sum of simple and non-simple commutators, i.e. we have (a) $\Rightarrow$ (b). $\square$

Lemma 5.4.

(a) *For every word w in the alphabet $\mathbb{L}=\{a,b,c,d,e,v\}$, containing at least two letters, and for any letters $s,s'\in\mathbb{L}$, $[ss'w]=[s'sw]$ holds in the quotient $\overline{L}_4$.*
(b) *Let w be any word in the alphabet $\mathbb{L}$, containing at least one letter d or e. Then in the quotient $\overline{L}_4$ the following relations hold*:

$$[adx]=[edx],\qquad [aex]=\big[e^2x\big],\qquad [awx]=[ewx];$$
$$[bdx]=\big[(-d-e)dx\big],\qquad [bex]=\big[(-d-e)ex\big],\qquad [bwx]=\big[(-d-e)wx\big];$$
$$[cdx]=\big[d^2x\big],\qquad [cex]=[dex],\quad [cwx]=[dwx].$$

Proof. (a) This is a direct analogue of Claim 2.6(a).
(b) Apply the item (a) and Claim 5.2 as follows:

$$[adx]\overset{(5.2c)}{=}[adu]\overset{(5.4a)}{=}[dau]\overset{(5.2c)}{=}[dex]\overset{(5.4a)}{=}[edx],$$
$$[aex]\overset{(5.4a)}{=}[eax]\overset{(5.2a)}{=}-[ebx]-[ecx]=\big[e^2x\big];$$
$$[bdx]\overset{(5.2c)}{=}[bdu]\overset{(5.4a)}{=}[dbu]\overset{(5.2c)}{=}[dvx]\overset{(5.2b)}{=}\big[d(-d-e)x\big]\overset{(5.4a)}{=}\big[(-d-e)dx\big];$$
$$[bex]\overset{(5.2c)}{=}-[bey]\overset{(5.4a)}{=}-[eby]\overset{(5.2c)}{=}[evx]\overset{(5.2b)}{=}\big[e(-d-e)x\big]\overset{(5.4a)}{=}\big[(-d-e)ex\big];$$
$$[cdx]\overset{(5.4a)}{=}[dcx]\overset{(5.2a)}{=}-[dax]-[dbx]\overset{(5.4b)}{=}\big[d^2x\big],$$
$$[cex]\overset{(5.2c)}{=}-[cey]\overset{(5.4a)}{=}-[ecy]\overset{(5.2c)}{=}[edx].$$

If a word w in $\mathbb{L}$ contains at least one letter d or e, then by the item (a) there is a word w' in $\mathbb{L}$, such that $[wx]=[w'dx]$ (without loss of generality) or $[wx]=[w'ex]$. Then

$$[awx]=\big[aw'dx\big]\overset{(5.4a)}{=}\big[w'adx\big]\overset{(5.4b)}{=}\big[w'edx\big]\overset{(5.4a)}{=}\big[ew'dx\big]=[ewx].$$

The proof in the other cases is similar. □

Lemma 5.5.

(a) *In each degree $n\geqslant 2$, every simple commutator $[w]\in\overline{L}_4$ is expressed linearly via $4(n-1)$ simple ones $[a^kb^lx]$, $[a^kd^ly]$, $[b^ke^lz]$, $[c^ke^lu]$, $k+l=n-2$, $k,l\geqslant 0$.*
(b) *In each degree $n\geqslant 3$, any non-simple commutator $[w]\in\overline{L}_4$ is expressed linearly via $n-1$ non-simple commutators $[d^ke^lx]$, where $k+l=n-2$, $k,l\geqslant 0$.*
(c) *In any degree $n\geqslant 3$, the quotient $\overline{L}_4$ is linearly generated by $4(n-1)$ simple commutators $[a^kb^lab]$, $[a^kd^lad]$, $[b^ke^lbe]$, $[c^ke^lce]$ and $n-1$ non-simple ones $[d^ke^lab]$, $k+l=n-2$, $k,l\geqslant 0$.*

Proof. (a) Let us consider a simple commutator $[w]$ not containing upper index 4. Then the commutator $[w]$ contains only the letters a,b,c. By the relations (5.2a) $[cx]=-[ax]-$

$[bx]$, one can express $[w]$ via the commutators $[a^k b^l x]$, $k, l \geqslant 0$. The proof is analogous for other three types of simple commutators.

(b) Let $w = w'' s_1 s_2 s_3$ be the given word, where s_1, s_2, s_3 are the three last letters of w. Since $[w]$ is non-simple, then by Lemma 5.4(a) one can permute the letters of w in such a way that the commutator $[s_1 s_2 s_3]$ becomes non-simple. By Claim 5.3(b) the commutator $[s_1 s_2 s_3]$ is expressed via $[dx]$ and $[ex]$. Then by Lemma 5.4(a) $[w] = [w'' s_1 s_2 s_3]$ can be rewritten in terms of $[dw''x]$ and $[ew''x]$. Hence one can apply the induction on the length of w.

Item (c) follows from (a) and (b). □

5.2. *Calculations in the quotient $\overline{L}_4$*

Claim 5.6. *For all $k \geqslant 0$, $l \geqslant 1$, in the quotient $\overline{L}_4$ the following relations hold*:

(a) $[b^k d^l x] = [(-d-e)^k d^l x]$, $[d^k b^l y] = -[d^k (-d-e)^l x]$;
(b) $[b^k c^l z] = [(-d-e)^k d^l x]$, $[c^k b^l u] = [d^k (-d-e)^l x]$;
(c) $[d^k e^l y] = -[d^k e^l x]$, $[e^k d^l u] = [e^k d^l x]$;
(d) $[a^k c^l y] = -[e^k d^l x]$, $[c^k a^l u] = [d^k e^l x]$.

Proof. (a) By Lemma 5.4(b) one has

$$\big[bd^l x\big] = \big[(-d-e)d^l x\big], \quad \text{i.e.}$$
$$\big[b^2 d^l x\big] = \Big[(-d-e)bd^l x\Big] = \big[(-d-e)^2 d^l x\big]$$

and so on, i.e.

$$\big[b^k d^l x\big] = \big[(-d-e)^k d^l x\big] \quad \text{holds for all } k \geqslant 0,\ l \geqslant 1.$$

Similarly,

$$\begin{aligned}
&[by] \overset{(5.2c)}{=} -[vx] \overset{(5.2b)}{=} -\big[(-d-e)x\big] \\
&\Rightarrow \quad \big[b^2 y\big] = -\big[b(-d-e)x\big] \overset{(5.4b)}{=} -\big[(-d-e)^2 x\big] \\
&\Rightarrow \quad \big[b^l y\big] = -\big[(-d-e)^l x\big] \\
&\Rightarrow \quad \big[d^k b^l y\big] = -\big[d^k (-d-e)^l x\big] \quad \text{for all } k \geqslant 0,\ l \geqslant 1.
\end{aligned}$$

The items (b)–(d) are proved analogously to (a). Apply the following formulae:

$$\begin{aligned}
[cz] \overset{(5.2c)}{=} -[dx] \quad &\Rightarrow \quad \big[c^2 z\big] = -[cdx] \overset{(5.4b)}{=} -\big[d^2 x\big] \\
&\Rightarrow \quad \big[c^l z\big] = -\big[d^l x\big] \\
&\Rightarrow \quad \big[b^k c^l z\big] = -\big[b^k d^l x\big] \overset{(5.6a)}{=} -\big[(-d-e)^k d^l x\big] \quad \text{for all } k \geqslant 0,\ l \geqslant 1.
\end{aligned}$$

$$[bu] \overset{(5.2c)}{=} [vx] \overset{(5.2b)}{=} \big[(-d-e)x\big]$$
$$\Rightarrow \quad \big[b^2u\big] = \big[b(-d-e)x\big] \overset{(5.4b)}{=} \big[(-d-e)^2x\big]$$
$$\Rightarrow \quad \big[b^lu\big] = \big[(-d-e)^lx\big]$$
$$\Rightarrow \quad \big[cb^lu\big] = \big[c(-d-e)^lx\big] \overset{(5.4b)}{=} \big[d(-d-e)^lx\big]$$
$$\Rightarrow \quad \big[c^kb^lu\big] = \big[d^k(-d-e)^lx\big] \quad \text{for all } k \geqslant 0,\ l \geqslant 1.$$
$$[ey] \overset{(5.2c)}{=} -[ex] \quad \Rightarrow \quad \big[e^2y\big] = -\big[e^2x\big] \quad \Rightarrow \quad \big[e^ly\big] = -\big[e^lx\big]$$
$$\Rightarrow \quad \big[d^ke^ly\big] = -\big[d^ke^lx\big], \quad k \geqslant 0,\ l \geqslant 0.$$
$$[du] \overset{(5.2c)}{=} [dx] \quad \Rightarrow \quad \big[d^lu\big] = \big[d^lx\big] \quad \Rightarrow \quad \big[e^kd^lu\big] = \big[e^kd^lx\big] \quad \text{for all } k \geqslant 0,\ l \geqslant 1.$$
$$[cy] \overset{(5.2c)}{=} -[dx] \quad \Rightarrow \quad \big[c^2y\big] = -[cdx] \overset{(5.4b)}{=} -\big[d^2x\big]$$
$$\Rightarrow \quad \big[c^ly\big] = -\big[d^lx\big] \quad \Rightarrow \quad \big[ac^ly\big] = -\big[ad^lx\big] \overset{(5.4b)}{=} -\big[ed^lx\big]$$
$$\Rightarrow \quad \big[a^kc^ly\big] = -\big[e^kd^lx\big] \quad \text{for all } k \geqslant 0,\ l \geqslant 1.$$

Finally, one has

$$[au] \overset{(5.2c)}{=} [ex] \quad \Rightarrow \quad \big[a^2u\big] = [aex] \overset{(5.4b)}{=} \big[e^2x\big] \quad \Rightarrow \quad \big[a^lu\big] = \big[e^lx\big]$$
$$\Rightarrow \quad \big[ca^lu\big] = \big[ce^lx\big] \overset{(5.4b)}{=} \big[de^lx\big] \quad \Rightarrow \quad \big[c^ka^lu\big] = \big[d^ke^lx\big]. \qquad \square$$

Claim 5.7. *For any $k \geqslant 0$, in the quotient $\overline{L}_4$ one has*:

(a) $[(b+d)^kx] = [b^kx] - [(-d-e)^kx] + [(-e)^kx]$, $[(b+d)^ky] = [d^ky] + [d^kx] - [(-e)^kx]$;
(b) $[(b+c)^kz] = [b^kz] + [(-d-e)^kx] - [(-e)^kx]$, $[(b+c)^ku] = [c^ku] - [d^kx] + [(-e)^kx]$;
(c) $[(d+e)^ky] = [d^ky] + [d^kx] - [(d+e)^kx]$, $[(d+e)^ku] = [e^ku] - [e^kx] + [(d+e)^kx]$;
(d) $[(a+c)^ky] = [a^ky] + [e^kx] - [(d+e)^kx]$, $[(a+c)^ku] = [c^ku] - [d^kx] + [(d+e)^kx]$.

Proof. (a) For each $k \geqslant 0$, one obtains

$$\big[(b+d)^kx\big] = \big[b^kx\big] + \left[\sum_{j=1}^{k}\binom{k}{j}b^{k-j}d^jx\right] \overset{(5.6a)}{=} \big[b^kx\big] + \left[\sum_{j=1}^{k}\binom{k}{j}(-d-e)^{k-j}d^jx\right]$$
$$= \big[b^kx\big] - \big[(-d-e)^kx\big] + \left[\sum_{j=0}^{k}\binom{k}{j}(-d-e)^{k-j}d^jx\right]$$
$$= \big[b^kx\big] - \big[(-d-e)^kx\big] + \big[(d-d-e)^kx\big],$$

$$\begin{aligned}
\big[(b+d)^k y\big] &= \big[d^k y\big] + \left[\sum_{j=0}^{k}\binom{k-1}{j} b^{k-j} d^j y\right] \\
&\overset{(5.6a)}{=} \big[d^k y\big] - \left[\sum_{j=0}^{k-1}\binom{k}{j}(-d-e)^{k-j} d^j x\right] \\
&= \big[d^k y\big] + \big[d^k x\big] - \left[\sum_{j=0}^{k}\binom{k}{j}(-d-e)^{k-j} d^j x\right] \\
&= \big[d^k y\big] + \big[d^k x\big] - \big[(-d-e+d)^k x\big].
\end{aligned}$$

Items (b)–(d) are proved analogously to (a). The following formulae are used:

$$\begin{aligned}
\big[(b+c)^k z\big] - \big[b^k z\big] &= \sum_{j=1}^{k}\binom{k}{j}\big[b^{k-j} c^j z\big] \overset{(5.6b)}{=} -\sum_{j=1}^{k}\binom{k}{j}\big[(-d-e)^{k-j} d^j x\big] \\
&= \big[(-d-e)^k x\big] - \big[(-e)^k x\big], \\
\big[(b+c)^k u\big] - \big[c^k u\big] &= \sum_{j=0}^{k-1}\binom{k}{j}\big[b^{k-j} c^j u\big] \overset{(5.6b)}{=} \sum_{j=0}^{k-1}\binom{k}{j}\big[(-d-e)^{k-j} d^j x\big] \\
&= -\big[d^k x\big] + \big[(-e)^k x\big], \\
\big[(d+e)^k y\big] - \big[d^k y\big] &= \sum_{j=1}^{k}\binom{k}{j}\big[d^{k-j} e^j y\big] \overset{(5.6c)}{=} -\sum_{j=1}^{k}\binom{k}{j}\big[d^{k-j} e^j x\big] \\
&= \big[d^k x\big] - \big[(d+e)^k x\big], \\
\big[(d+e)^k u\big] - \big[e^k u\big] &= \sum_{j=1}^{k}\binom{k}{j}\big[e^{k-j} d^j u\big] \overset{(5.6c)}{=} \sum_{j=1}^{k}\binom{k}{j}\big[e^{k-j} d^j x\big] \\
&= -\big[e^k x\big] + \big[(d+e)^k x\big], \\
\big[(a+c)^k y\big] - \big[a^k y\big] &= \sum_{j=1}^{k}\binom{k}{j}\big[a^{k-j} c^j y\big] \overset{(5.6d)}{=} -\sum_{j=1}^{k}\binom{k}{j}\big[e^{k-j} d^j x\big] \\
&= \big[e^k x\big] - \big[(d+e)^k x\big], \\
\big[(a+c)^k u\big] - \big[c^k u\big] &= \sum_{j=1}^{k}\binom{k}{j}\big[c^{k-j} a^j y\big] \overset{(5.6d)}{=} \sum_{j=1}^{k}\binom{k}{j}\big[d^{k-j} e^j x\big] \\
&= -\big[d^k x\big] + \big[(d+e)^k x\big]. \qquad \square
\end{aligned}$$

Lemma 5.8.

(a) *For all* $k, l \geqslant 0$, *in the quotient* $\overline{L}_4$ *one gets*

$$\begin{cases} [a^k(b+d)^l a(b+d)] = ([a^k b^l x] + [a^k d^l y]) + [e^k d^l x] - [e^k(-d-e)^l x], \\ [(b+c)^k e^l (b+c)e] = ([b^k e^l z] + [c^k e^l u]) - [d^k e^l x] + [(-d-e)^k e^l x]. \end{cases} \tag{5.8a}$$

(b) *For all* $k, l \geqslant 0$, *in the quotient* $\overline{L}_4$ *one has*

$$\big[(a+c)^k(d+e)^l(a+c)(d+e)\big] = \big(\big[a^k d^l y\big] + \big[c^k e^l u\big]\big) + \big[e^k d^l x\big] - \big[d^k e^l x\big]. \tag{5.8b}$$

Proof. (a) One has $[a, b+d] \overset{(5.1a)}{=} x + y$, $[b+c, e] \overset{(5.1a)}{=} z + u$. For all $k \geqslant 0$, $l \geqslant 1$, one gets

$$\begin{aligned} \big[a^k(b+d)^l a(b+d)\big] &= \big[a^k(b+d)^l x\big] + \big[a^k(b+d)^l y\big] \\ &\overset{(5.7a)}{=} \big(\big[a^k b^l x\big] - \big[a^k(-d-e)^l x\big] + \big[a^k(-e)^l x\big]\big) \\ &\quad + \big(\big[a^k d^l y\big] + \big[a^k d^l x\big] - \big[a^k(-e)^l x\big]\big) \\ &\overset{(5.4b)}{=} \big(\big[a^k b^l x\big] + \big[a^k d^l y\big]\big) + \big[e^k d^l x\big] - \big[e^k(-d-e)^l x\big]. \end{aligned}$$

Observe that the above equations hold for $l = 0$ also. Similarly, one gets

$$\begin{aligned} \big[(b+c)^k e^l (b+c)e\big] &\overset{(5.4a)}{=} \big[e^l(b+c)^k z\big] + \big[e^l(b+c)^k u\big] \\ &\overset{(5.7b)}{=} \big(\big[e^l b^k z\big] + \big[e^l(-d-e)^k x\big] - \big[e^l(-e)^k x\big]\big) \\ &\quad + \big(\big[e^l c^k u\big] - \big[e^l d^k x\big] + \big[e^l(-e)^k x\big]\big) \\ &= \big(\big[e^l b^k z\big] + \big[e^l c^k u\big]\big) - \big[e^l d^k x\big] + \big[e^l(-d-e)^k x\big]. \end{aligned}$$

(b) Analogously to the item (a), for all $k, l \geqslant 0$, one obtains

$$\begin{aligned} &\big[(a+c)^k(d+e)^l(a+c)(d+e)\big] \\ &\quad \overset{(5.1a)}{=} \big[(a+c)^k(d+e)^l y\big] + \big[(a+c)^k(d+e)^l u\big] \\ &\quad \overset{(5.7c)}{=} \big[(a+c)^k d^l y\big] + \big[(a+c)^k d^l x\big] - \big[(a+c)^k(d+e)^l x\big] + \big[(a+c)^k e^l u\big] \\ &\qquad - \big[(a+c)^k e^l x\big] + \big[(a+c)^k(d+e)^l x\big] \\ &\quad \overset{(5.2a),(5.4a)}{=} \big[d^l(a+c)^k y\big] + \big[d^l(-b)^k x\big] + \big[e^l(a+c)^k u\big] - \big[e^l(-b)^k x\big] \\ &\quad \overset{(5.7d),(5.4a)}{=} \big(\big[d^l a^k y\big] + \big[d^l e^k x\big] - \big[d^l(d+e)^k x\big]\big) \end{aligned}$$

$$+\big([e^l c^k u]-[e^l d^k x]+[e^l (d+e)^k x]\big)+\big[(-b)^k\big(d^l-e^l\big)x\big]$$

$$\overset{(5.4a),(5.4b)}{=}\big([a^k d^l y]+[c^k e^l u]\big)+[e^k d^l x]-[d^k e^l x]+\big[\big(e^l-d^l\big)(d+e)^k x\big]$$

$$+\big[(d+e)^k\big(d^l-e^l\big)x\big]$$

$$\Big(\big[(-b)^k(d^l-e^l)x\big]\overset{(5.4b)}{=}\big[(d+e)^k(d^l-e^l)x\big]\text{ was used}\Big)$$

$$=\big([a^k d^l y]+[c^k e^l u]\big)+[e^k d^l x]-[d^k e^l x].$$

Note that the relation $[(-b)^k(d^l-e^l)x]\overset{(5.4b)}{=}[(d+e)^k(d^l-e^l)x]$ holds for any $l\geqslant 0$. □

5.3. Checking the compressed pentagon $\overline{(1.3c)}$

Lemma 5.9. *For any compressed associator $\bar\varphi\in\widehat{\overline{L}}_3$, the compressed pentagon $\overline{(1.3c)}$ is equivalent to the following equation in the algebra $\widehat{\overline{L}}_4$:*

$$\bar\varphi(b,e)+\bar\varphi(a+c,d+e)+\bar\varphi(a,b)=\bar\varphi(a,b+d)+\bar\varphi(b+c,e). \tag{5.9}$$

Proof. Let us rewrite explicitly the pentagon (1.3c) for a compressed associator $\bar\varphi\in\widehat{\overline{L}}_3$:

$$\exp\big(\bar\varphi(b,e)\big)\cdot\exp\big(\bar\varphi(a+c,d+e)\big)\cdot\exp\big(\bar\varphi(a,b)\big)=\exp\big(\bar\varphi(a,b+d)\big)\cdot\exp\big(\bar\varphi(b+c,e)\big).$$

Since in the quotient $\overline{L}_4$ all commutators commute, then the series $\bar\varphi(b,e)$, $\bar\varphi(a+c,d+e)$, $\bar\varphi(a,b)$, $\bar\varphi(a,b+d)$, and $\bar\varphi(b+c,e)$ commute with each other in $\widehat{\overline{L}}_4$. Hence, taking the logarithm of both sides of the above pentagon, one needs to apply the simplest case of CBH formula (2.3): $\log(\exp(P)\cdot\exp(Q))=P+Q$ provided that P,Q commute. □

Proposition 5.10. *Let $f(\lambda,\mu)=\sum_{k,l\geqslant 0}\alpha_{kl}\lambda^k\mu^l$ be the generating function of the coefficients $\alpha_{kl}=\alpha_{lk}$ of a compressed Drinfeld associator $\bar\varphi\in\widehat{\overline{L}}_3$. Then in the algebra $\widehat{\overline{L}}_4$ the compressed pentagon equation* $\overline{(1.3c)}=(5.9)$ *follows from the symmetry $\alpha_{kl}=\alpha_{lk}$.*

Proof. By Lemma 5.8(b) the left-hand side of (5.9) is

$$\sum_{k,l\geqslant 0}\alpha_{kl}\big([b^k e^l be]+[(a+c)^k(d+e)^l(a+c)(d+e)]+[a^k b^l ab]\big)$$

$$\overset{(5.8b)}{=}\sum_{k,l\geqslant 0}\alpha_{kl}\big([b^k e^l z]+\big([a^k d^l y]+[c^k e^l u]+[e^k d^l x]-[d^k e^l x]\big)+[a^k b^l x]\big).$$

Similarly by Lemma 5.8(a) the right-hand side of (5.9) is

$$\sum_{k,l\geqslant 0}\alpha_{kl}\big([a^k(b+d)^k a(b+d)]+[(b+c)^k e(b+c)e]\big)$$

$$\overset{(5.8a)}{=} \sum_{k,l\geqslant 0} \alpha_{kl}\Big(\big([a^k b^l x] + [a^k d^l y] + [e^k d^l x] - [e^k(-d-e)^l x]\big) + \big([b^k e^l z] + [c^k e^l u] - [d^k e^l x] + [(-d-e)^k e^l x]\big)\Big).$$

The difference is

$$\sum_{k,l\geqslant 0} \alpha_{kl}\big([e^k(-d-e)^l x] - [(-d-e)^k e^l x]\big) \overset{(5.4a)}{=} 0 \quad \text{if } \alpha_{kl} = \alpha_{lk}. \qquad \square$$

The pentagon part of Theorem 1.5(c) follows from Proposition 5.10.

6. Drinfeld series, zeta values, and problems

In this section one shall check that the Drinfeld series $f^{\mathrm{D}}(\lambda,\mu)$ from Definition 6.1(b) (a compressed associator expressed via zeta values) is contained in the general family (1.5c).

6.1. Riemann zeta-function of even integers

Definition 6.1 (Riemann zeta function $\zeta(n)$, the Drinfeld series $s(\lambda,\mu)$).

(a) Let $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$ be *the classical Riemann zeta function*. Put

$$\theta_n := \frac{\zeta(n)}{n(\pi\sqrt{-1})^n}.$$

(b) Set

$$S(\lambda) = \sum_{n=2}^{\infty} \theta_n \lambda^n = \sum_{n=2}^{\infty} \frac{\zeta(n)\lambda^n}{n(\pi\sqrt{-1})^n}$$

and

$$s(\lambda,\mu) = S(\lambda) + S(\mu) - S(\lambda+\mu) = \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} \cdot \frac{\lambda^n + \mu^n - (\lambda+\mu)^n}{(\pi\sqrt{-1})^n}.$$

Due to the change $t^{ij} \mapsto 2t^{ij}$ in the hexagon (1.3b), 2^n is missing in the denominators of $s(\lambda,\mu)$. The following theorem is quoted from [12, Chapter XIX, Remark 6.6(b), p. 468].

Theorem 6.2 [10]. *There is a compressed Drinfeld associator*

$$\bar{\varphi}^{\mathrm{D}}(a,b) = \sum_{k,l\geqslant 0} \alpha_{kl}^{\mathrm{D}}[a^k b^l ab]$$

defined by the Drinfeld series

$$f^{\mathrm{D}}(\lambda,\mu)=\sum_{k,l\geqslant 0}\alpha^{\mathrm{D}}_{kl}\lambda^k\mu^l,$$

where

$$\tilde{f}^{\mathrm{D}}(\lambda,\mu)=1+\lambda\mu f^{\mathrm{D}}(\lambda,\mu)=\exp\bigl(s(\lambda,\mu)\bigr). \tag{6.2}$$

The Drinfeld series $s(\lambda,\mu)$ leads to the well-known formula for even zeta values [8,17].

Lemma 6.3. *For each $n\geqslant 1$, one has*

$$2n\theta_{2n}=-\frac{2^{2n}B_{2n}}{2(2n)!}\quad\textit{and}\quad \zeta(2n)=(-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}.$$

Proof. Lemma 4.2 says that

$$f(\lambda,-\lambda)=\frac{1}{\lambda^2}-\frac{2}{\lambda(e^{\lambda}-e^{-\lambda})}.$$

By Definition 6.1(b) one has

$$s(\lambda,-\lambda)=2\sum_{n=1}^{\infty}\theta_{2n}\lambda^{2n}.$$

Let us substitute $\mu=-\lambda$ in formula (6.2):

$$2\sum_{n=1}^{\infty}\theta_{2n}\lambda^{2n}=\log\bigl(\tilde{f}(\lambda,-\lambda)\bigr)=\log\bigl(1-\lambda^2 f(\lambda,-\lambda)\bigr)=\log\biggl(\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}\biggr). \tag{6.3}$$

Taking the first derivative of the above equation with respect to λ, one obtains

$$\biggl(\frac{e^{\lambda}-e^{-\lambda}-\lambda(e^{\lambda}+e^{-\lambda})}{(e^{\lambda}-e^{-\lambda})^2}\biggr):\biggl(\frac{\lambda}{e^{\lambda}-e^{-\lambda}}\biggr)=\frac{1}{\lambda}-1-\frac{2}{e^{2\lambda}-1}=2\sum_{n=1}^{\infty}(2n\theta_{2n})\lambda^{2n-1}.$$

Multiply the resulting equation by λ and use the definition of the Bernoulli numbers:

$$2\sum_{n=1}^{\infty}(2n\theta_{2n})\lambda^{2n}=1-\lambda-\frac{2\lambda}{e^{2\lambda}-1}=-\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(2\lambda)^{2n}.$$

Hence

$$2n\theta_{2n}=-\frac{B_{2n}}{2(2n)!}2^{2n}.$$

It remains to use the formula from Definition 6.1(a):

$$\theta_{2n} = \frac{\zeta(2n)}{2n(\pi\sqrt{-1})^{2n}} = (-1)^n \frac{\zeta(2n)}{2n\pi^{2n}}. \qquad \square$$

Example 6.4. By Lemma 6.3 and Table A.2 one can easily calculate:

$$\theta_2 = -\frac{1}{12}, \quad \theta_4 = \frac{1}{360}, \quad \theta_6 = -\frac{1}{5670}, \quad \theta_8 = \frac{1}{75600}, \quad \theta_{10} = -\frac{1}{935550};$$
$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(10) = \frac{\pi^{10}}{93555}.$$

6.2. Odd zeta values can be considered as free parameters in Theorem 6.2

Claim 6.5. *Any series $s(\lambda,\mu)$ obeys the following relations*:

$$Even\big(\exp\big(s(\lambda,\mu)\big)\big) = \exp\big(Even\big(s(\lambda,\mu)\big)\big) \cdot Even\big(\exp\big(Odd\big(s(\lambda,\mu)\big)\big)\big), \tag{6.5a}$$
$$Odd\big(\exp\big(s(\lambda,\mu)\big)\big) = \exp\big(Even\big(s(\lambda,\mu)\big)\big) \cdot Odd\big(\exp\big(Odd\big(s(\lambda,\mu)\big)\big)\big). \tag{6.5b}$$

Proof. Actually, one has

$$\begin{aligned}
&Even\big(\exp\big(s(\lambda,\mu)\big)\big)\\
&= \frac{\exp(s(\lambda,\mu)) + \exp(s(-\lambda,-\mu))}{2}\\
&= \frac{\exp(Even(s(\lambda,\mu)) + Odd(s(\lambda,\mu))) + \exp(Even(s(\lambda,\mu)) - Odd(s(\lambda,\mu)))}{2}\\
&= \exp\big(Even\big(s(\lambda,\mu)\big)\big) \cdot \frac{\exp(Odd(s(\lambda,\mu))) + \exp(-Odd(s(\lambda,\mu)))}{2}\\
&\Leftrightarrow \quad (6.5a).
\end{aligned}$$

Formula (6.5b) is proved absolutely analogously. $\square$

Claim 6.6.

(a) *For the series $S(\rho)$, one has*

$$\exp\big(-2Even\big(S(\rho)\big)\big) = \frac{e^{\rho} - e^{-\rho}}{2\rho}. \tag{6.6a}$$

(b) *For the Drinfeld series $s(\lambda,\mu)$, one has*

$$\exp\big(Even\big(s(\lambda,\mu)\big)\big) = \sqrt{\frac{e^{\lambda+\mu} - e^{-\lambda-\mu}}{2(\lambda+\mu)} \cdot \frac{2\lambda}{e^{\lambda} - e^{-\lambda}} \cdot \frac{2\mu}{e^{\mu} - e^{-\mu}}}. \tag{6.6b}$$

Proof. (a) By Definition 6.1(b), one obtains $Even(S(\rho)) = \sum_{n=1}^{\infty} \theta_{2n}\rho^{2n}$. So, one needs to get

$$-2\sum_{n=1}^{\infty}\theta_{2n}\rho^{2n} = \log\left(\frac{e^{\rho}-e^{-\rho}}{2\rho}\right).$$

This follows immediately from formula (6.3).

(b) Apply the item (a) and Definition 6.1(b):

$$\begin{aligned}
\exp\bigl(Even\bigl(s(\lambda,\mu)\bigr)\bigr) &\overset{(6.1b)}{=} \exp\bigl(Even\bigl(S(\lambda)\bigr)\bigr)\cdot\exp\bigl(Even\bigl(S(\mu)\bigr)\bigr)\cdot\exp\bigl(-Even\bigl(S(\lambda+\mu)\bigr)\bigr)\\
&\overset{(6.6a)}{=} \sqrt{\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}}\sqrt{\frac{2\mu}{e^{\mu}-e^{-\mu}}}\sqrt{\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}}\\
&= \sqrt{\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}\cdot\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}\cdot\frac{2\mu}{e^{\mu}-e^{-\mu}}}. \qquad \square
\end{aligned}$$

Lemma 6.7. *Take any values* $\zeta(2n+1)\in\mathbb{C}$ *and put*

$$\theta_{2n+1} = \frac{(-1)^n\zeta(2n+1)}{(2n+1)\pi^{2n+1}\sqrt{-1}}.$$

Set

$$\begin{aligned}
\theta(\lambda,\mu) &= -\sum_{n=1}^{\infty}\theta_{2n+1}F^{\mathrm{D}}_{2n+1}(\lambda,\mu)\\
&= \sum_{n=1}^{\infty}\frac{(-1)^n\zeta(2n+1)}{(2n+1)\pi^{2n+1}\sqrt{-1}}\bigl(\lambda^{2n+1}+\mu^{2n+1}-(\lambda+\mu)^{2n+1}\bigr).
\end{aligned}$$

Then there exist series

$$h(\lambda,\mu) := \left(\sum_{k=0}^{\infty}\frac{(\theta(\lambda,\mu))^{2k}}{(2k)!}\right) : \sqrt{\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}\cdot\frac{e^{\lambda}-e^{-\lambda}}{2\lambda}\cdot\frac{e^{\mu}-e^{-\mu}}{2\mu}}, \tag{6.7a}$$

$$\tilde{h}(\lambda,\mu) := \left(\sum_{k=0}^{\infty}\frac{\bigl(\theta(\lambda,\mu)\bigr)^{2k+1}}{(2k+1)!}\right) : \lambda\mu(\lambda+\mu)\sqrt{\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}\cdot\frac{e^{\lambda}-e^{-\lambda}}{2\lambda}\cdot\frac{e^{\mu}-e^{-\mu}}{2\mu}}, \tag{6.7b}$$

such that, for some coefficients $\beta_{nk}, \tilde{\beta}_{nk}\in\mathbb{C}$*, the series* $h(\lambda,\mu), \tilde{h}(\lambda,\mu)$ *have the form*

$$h(\lambda,\mu)=\frac{2\omega}{e^{\omega}-e^{-\omega}}+\sum_{n=3}^{\infty}\sum_{k=1}^{[n/3]}\beta_{nk}\lambda^{2k}\mu^{2k}(\lambda+\mu)^{2k}\left(\lambda^2+\lambda\mu+\mu^2\right)^{n-3k},$$

$$\tilde{h}(\lambda,\mu)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}\tilde{\beta}_{nk}\lambda^{2k}\mu^{2k}(\lambda+\mu)^{2k}\left(\lambda^2+\lambda\mu+\mu^2\right)^{n-3k},$$

$$\omega=\sqrt{\lambda^2+\lambda\mu+\mu^2}.$$

In other words, the coefficients β_{nk}, $\tilde{\beta}_{nk}$ of the series $h(\lambda,\mu)$, $\tilde{h}(\lambda,\mu)$ can be expressed via θ_{2n+1} in such a way that Eqs. (6.7a) *and* (6.7b) *hold identically.*

Proof. Associator polynomials form an algebra. The right-hand side $R_a(\lambda,\mu)$ of (6.7a) is a well-defined series since both factors are units. Also $R_a(\lambda,\mu)$ is unchanged under the transformations $\lambda\leftrightarrow\mu$ and $\mu\leftrightarrow(-\lambda-\mu)$. Lemma 4.9 and Proposition 4.10(a) imply

$$h(\lambda,\mu)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}\beta_{nk}\lambda^{2k}\mu^{2k}(\lambda+\mu)^{2k}\left(\lambda^2+\lambda\mu+\mu^2\right)^{n-3k}$$

for some $\beta_{nk}\in\mathbb{C}$, $n\geqslant 3k\geqslant 0$. When $\lambda+\mu=0$ one gets

$$h(\lambda,-\lambda)=\left(\sum_{k=0}^{\infty}\frac{(\theta(\lambda,-\lambda))^{2k}}{(2k)!}\right):\sqrt{1\cdot\frac{e^{\lambda}-e^{-\lambda}}{2\lambda}\cdot\frac{e^{-\lambda}-e^{\lambda}}{-2\lambda}}=\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}.$$

Hence

$$\sum_{n=0}^{\infty}\beta_{n0}\left(\lambda^2+\lambda\mu+\mu^2\right)^{n}=\frac{2\omega}{e^{\omega}-e^{-\omega}}.$$

From Eq. (6.7a) the other coefficients β_{nk} can be expressed by even degree monomials in θ_{2n+1}.

Similarly, the right-hand side $R_b(\lambda,\mu)$ of (6.7b) is well defined since both factors are units multiplied by $\lambda\mu(\lambda+\mu)$. Also $R_b(\lambda,\mu)$ has the symmetries $R_b(\lambda,\mu)=R_b(\mu,\lambda)=R_b(\lambda,-\lambda-\mu)$. By Lemma 4.9 and Proposition 4.10(b) one has

$$\tilde{h}(\lambda,\mu)=\sum_{n=1}^{\infty}\sum_{k=0}^{[(n-1)/3]}\tilde{\beta}_{nk}\lambda^{2k+1}\mu^{2k+1}(\lambda+\mu)^{2k+1}\left(\lambda^2+\lambda\mu+\mu^2\right)^{n-3k}$$

for some $\tilde{\beta}_{nk}\in\mathbb{C}$. From (6.7b) the coefficients $\tilde{\beta}_{nk}$ can be expressed via odd degree monomials in θ_{2n+1}. □

Claim 6.8. *Equations* (6.7a) *and* (6.7b) *are equivalent to* (6.8a) *and* (6.8b), *respectively*:

$$\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}\left(\frac{2\omega}{e^{\omega}-e^{-\omega}}+\sum_{n=3}^{\infty}h_n(\lambda,\mu)\right)=Even\bigl(\exp\bigl(s(\lambda,\mu)\bigr)\bigr),\quad and \tag{6.8a}$$

$$\lambda\mu\cdot\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2}\left(\sum_{n=0}^{\infty}\tilde{\beta}_{n0}\omega^{2n}+\sum_{n=3}^{\infty}\tilde{h}_n(\lambda,\mu)\right)=Odd\bigl(\exp\bigl(s(\lambda,\mu)\bigr)\bigr), \tag{6.8b}$$

where

$$h_n(\lambda,\mu)=\sum_{k=1}^{[n/3]}\beta_{nk}\lambda^{2k}\mu^{2k}(\lambda+\mu)^{2k}\omega^{2n-6k}\quad for\ n\geqslant 3,\ \omega=\sqrt{\lambda^2+\lambda\mu+\mu^2}.$$

The polynomials $\tilde{h}_n(\lambda,\mu)$ are defined by the same formula as $h_n(\lambda,\mu)$, except the coefficients $\tilde{\beta}_{nk}\in\mathbb{C}$ are substituted for $\beta_{nk}\in\mathbb{C}$.

Proof. It follows from Claims 6.5, 6.6, the formulae $Odd(s(\lambda,\mu))=\theta(\lambda,\mu)$, and

$$Even\bigl(\exp\bigl(Odd\bigl(s(\lambda,\mu)\bigr)\bigr)\bigr)=\sum_{k=0}^{\infty}\frac{\theta^{2k}(\lambda,\mu)}{(2k)!},$$

$$Odd\bigl(\exp\bigl(Odd\bigl(s(\lambda,\mu)\bigr)\bigr)\bigr)=\sum_{k=0}^{\infty}\frac{\theta^{2k+1}(\lambda,\mu)}{(2k+1)!}.\qquad\square$$

Proposition 6.9. *In Theorem* 6.2 *odd zeta values $\zeta(2n+1)$ can be considered as free parameters*: *for any values $\zeta(2n+1)\in\mathbb{C}$, the series $f^{\mathrm{D}}(\lambda,\mu)$ is a compressed associator.*

Proof. Starting with arbitrary values $\zeta(2n+1)\in\mathbb{C}$ Lemma 6.7 provides the parameters $\beta_{nk},\tilde{\beta}_{nk}$ such that Eqs. (6.8a) and (6.8b) hold identically. By Theorem 1.5(c) the series $s(\lambda,\mu)$ from Claim 6.8 gives rise to the generating series

$$f^{\mathrm{D}}(\lambda,\mu)=\frac{\exp(s(\lambda,\mu))-1}{\lambda\mu}.\qquad\square$$

Proposition 6.9 supports the long standing conjecture in the number theory: odd zeta values are algebraically independent over the rationals [8]. Proposition 6.9 will be reproved explicitly up to degree 7 in Claim A.6.

Proof of Corollaries 1.6(a), (b). (a) The first series $f^{\mathrm{I}}(\lambda,\mu)$ is obtained from the general formula (1.5c) by taking $\beta_{nk}=\tilde{\beta}_{nk}=0$. This solution is related with the associator polynomials $F^{\mathrm{I}}_{2n}(\lambda,\mu)=(\lambda^2+\lambda\mu+\mu^2)^n$.

(b) The second series $f^{\mathrm{II}}(\lambda,\mu)$ appears due to the associator polynomials

$$F^{\mathrm{II}}_{2n}(\lambda,\mu)=\frac{(\lambda+\mu)^{2n}+\lambda^{2n}+\mu^{2n}}{2},\quad n>1.$$

Actually, let γ_k be the coefficients of the series

$$\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}=\sum_{k=0}^{\infty}\gamma_k\lambda^{2k},$$

see (4.3). Then the series $1+\sum_{k=1}^{\infty}\gamma_k F^{\mathrm{II}}_{2k}(\lambda,\mu)$ plays the role of the function $h(\lambda,\mu)$ from Lemma 4.6. It remains to compute:

$$\begin{aligned}
1+\lambda\mu\cdot f^{\mathrm{II}}(\lambda,\mu)&=\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}\left(1+\sum_{k=1}^{\infty}\gamma_k\frac{(\lambda+\mu)^{2n}+\lambda^{2n}+\mu^{2n}}{2}\right)\\
&=\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}\cdot\frac{1}{2}\left(\frac{2(\lambda+\mu)}{e^{\lambda+\mu}-e^{-\lambda-\mu}}+\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}+\frac{2\mu}{e^{\mu}-e^{-\mu}}-1\right)\\
&=\frac{1}{2}+\frac{1}{2}\cdot\frac{e^{\lambda+\mu}-e^{-\lambda-\mu}}{2(\lambda+\mu)}\left(\frac{2\lambda}{e^{\lambda}-e^{-\lambda}}+\frac{2\mu}{e^{\mu}-e^{-\mu}}-1\right)\\
&\Leftrightarrow\quad(1.6\mathrm{b}).\qquad\square
\end{aligned}$$

Proof of Corollary 1.6(c). The proof follows from Proposition 6.9 since all odd zeta values $\zeta(2n+1)$ can be considered as free parameters in the Drinfeld series

$$f^{\mathrm{D}}(\lambda,\mu)=\frac{\exp(s(\lambda,\mu))-1}{\lambda\mu}.$$

The associator (1.6c) is obtained from $f^{\mathrm{D}}(\lambda,\mu)$ by using Lemma 6.3 and $\zeta(2k+1)=0$, $k\geqslant 1$. □

6.3. Conjectures and open problems

Theorem 1.5 describes only compressed associators. This is a first step in the general problem to find a complete rational associator. Theorem 1.5(c) gives a hope to describe Drinfeld associators up to triple commutators.

Problem 6.10. (a) Is it true that any compressed Drinfeld associator is the projection under $\widehat{L}_3\to[[\widehat{L}_3,\widehat{L}_3],[\widehat{L}_3,\widehat{L}_3]]$ of the logarithm $\varphi(a,b)$ of an honest one from Definition 1.3(b)?

(b) Describe all Drinfeld associators up to triple commutators. In other words, solve the hexagon and pentagon in the quotient $L_3/[L_3',[L_3',L_3']]$, where $L_3'=[L_3,L_3]$.

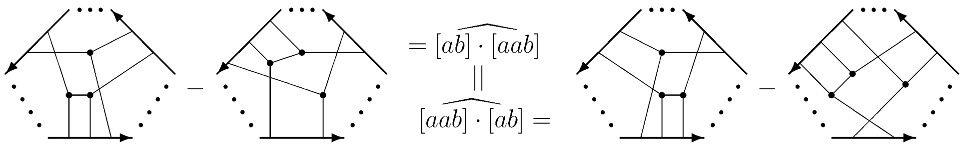


Fig. 2.

A compressed associator $\bar{\varphi}(a,b)$ already contains a lot of information. One may try to pass through the LM–BN construction [3,16] to get a well-defined invariant of knots in a quotient of the algebra A of chord diagrams. One wants to factorize A in such a way that the LM–BN construction leads to a knot invariant in this quotient. Roughly speaking, the LM–BN construction maps chord diagrams on n vertical strands onto chord diagrams on the circle. Let Σ be a sum of commutators in the symbols a,b,c. Define *the closure* $\widehat{\Sigma}=0$ as the relation $\Sigma=0$ in the algebra A of chord diagrams. Formally, one draws the relation $\Sigma=0$ on 3 vertical strands and assume that these strands are three arcs of a circle. For instance, the $4T$ relation from Definition 1.1(b) is the closure of $\Sigma=[a,b]-[b,c]$. Since $[abab]=[baab]$ holds in L_3, then at the stage of chord diagrams the first non-trivial relation is the closure of

$$\big[[ab],[aab]\big]=[ab]\cdot[aab]-[aab]\cdot[ab].$$

In Fig. 2 the relation was drawn briefly by using STU relations from [2].

After Vassiliev's paper [22] one usually uses another definition of Vassiliev invariants via chord diagrams [2]. A Vassiliev invariant of framed knots is the composition of the Kontsevich integral and *a weight function* on A, i.e. a linear function on chord diagrams, satisfying the $4T$ relations, see Definition 1.1(b).

Definition 6.11 (Compressed algebra $\overline{A}$ of chord diagrams, compressed Vassiliev invariants).

(a) For a word w_{kl} containing exactly k letters a and exactly l letters b, put

$$\Sigma(w_{kl}):=[w_{kl}ab]-\big[a^kb^lab\big],\quad k,l\geqslant 1.$$

Let $\overline{A}$ be the quotient of the classical algebra A of chord diagrams on the circle by the ideal generated by the relations $\widehat{\Sigma(w_{kl})}=0$ for all $k,l\geqslant 1$.

(b) *A compressed weight function* is a linear function on the compressed algebra $\overline{A}$. In other words, a Vassiliev invariant is *compressed*, if the corresponding weight function satisfies $\widehat{\Sigma(w_{kl})}=0$ for all $k,l\geqslant 1$. *The compressed Kontsevich integral* $\overline{Z}_K$ of a knot K is the image of the classical Kontsevich integral Z_K under the natural projection $A\to\overline{A}$.

Vassiliev invariants of degrees 2, 3, 4 are compressed ones, i.e. the theory is not empty.

Problem 6.12. (a) Check carefully that the LM–BN construction [3,16], for a compressed Drinfeld associator, gives rise to a well-defined knot invariant in the compressed algebra $\overline{A}$. Does the resulting invariant depend on a particularly chosen compressed associator?

(b) Which quantum invariants are compressed Vassiliev invariants?

(c) Describe all compressed Vassiliev invariants (as linear functions on the algebra $\overline{A}$).

(d) Compute the compressed Kontsevich integral for non-trivial knots, e.g., torus knots.

(e) Which knots can be classified via compressed Vassiliev invariants?

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Appendix A. Explicit formulae

Claim A.1. *For each $n \geqslant 1$, the following formulae hold*:

$$C_{2,2n} = C_{2n,2} = B_{2n},$$
$$C_{2,2n+1} = -C_{2n+1,2} = 2B_{2n+2},$$
$$C_{3,2n} = -C_{2n,3} = 3B_{2n+2} + B_{2n},$$
$$C_{3,2n+1} = C_{2n+1,3} = 3B_{2n+2},$$
$$C_{4,2n} = C_{2n,4} = 6B_{2n+2} + B_{2n},$$
$$C_{4,2n+1} = -C_{2n+1,4} = 4B_{2n+4} + 4B_{2n+2},$$
$$C_{5,2n} = -C_{2n,5} = 5B_{2n+4} + 10B_{2n+2} + B_{2n},$$
$$C_{5,2n+1} = C_{2n+1,5} = 10B_{2n+4} + 5B_{2n+2},$$
$$C_{6,2n} = C_{2n,6} = 15B_{2n+4} + 15B_{2n+2} + B_{2n},$$
$$C_{6,2n+1} = -C_{2n+1,6} = 6B_{2n+6} + 20B_{2n+4} + 6B_{2n+2}.$$

Proof. It follows from Lemma 2.11 and *the first key point*: $B_{2n+1} = 0$ for each $n \geqslant 1$. □

By Claim A.1 one can easily compute the numbers C_{mn} for $m + n \leqslant 12$.

By Table A.2 one can calculate the function $C(\lambda, \mu)$ up to degree 10, see Definition 2.4(b).

Example A.3. Up to degree 10 one has

Table A.2
Extended Bernoulli numbers C_{mn}

$m\backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0
2	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{15}$	$-\frac{1}{30}$	$\frac{1}{21}$	$\frac{1}{42}$	$-\frac{1}{15}$	$-\frac{1}{30}$	$\frac{5}{33}$	$\frac{5}{66}$	
3	0	$\frac{1}{15}$	$-\frac{1}{10}$	$\frac{4}{105}$	$\frac{1}{14}$	$-\frac{8}{105}$	$-\frac{1}{10}$	$\frac{32}{165}$	$\frac{5}{22}$		
4	$\frac{1}{30}$	$-\frac{1}{30}$	$-\frac{4}{105}$	$\frac{23}{210}$	$-\frac{4}{105}$	$-\frac{37}{210}$	$\frac{28}{165}$	$\frac{139}{330}$			
5	0	$-\frac{1}{21}$	$\frac{1}{14}$	$\frac{4}{105}$	$-\frac{3}{14}$	$\frac{16}{231}$	$\frac{13}{22}$				
6	$-\frac{1}{42}$	$\frac{1}{42}$	$\frac{8}{105}$	$-\frac{37}{210}$	$-\frac{16}{231}$	$\frac{305}{462}$					
7	0	$\frac{1}{15}$	$-\frac{1}{10}$	$-\frac{28}{165}$	$\frac{13}{22}$						
8	$\frac{1}{30}$	$-\frac{1}{30}$	$-\frac{32}{165}$	$\frac{139}{330}$							
9	0	$-\frac{5}{33}$	$\frac{5}{22}$								
10	$-\frac{5}{66}$	$\frac{5}{66}$									
11	0										

$$
\begin{aligned}
C(\lambda,\mu) = {} & -\frac{1}{2} + \frac{1}{12}(\lambda-\mu) + \frac{1}{24}\lambda\mu \\
& + \left(\frac{1}{720}\mu^3 + \frac{1}{180}\lambda\mu^2 - \frac{1}{180}\lambda^2\mu - \frac{1}{720}\lambda^3\right) \\
& - \left(\frac{1}{1440}\lambda^3\mu + \frac{1}{360}\lambda^2\mu^2 + \frac{1}{1440}\lambda\mu^3\right) \\
& + \frac{1}{7!}\left(\frac{\lambda^5}{6} + \lambda^4\mu + \frac{4}{3}\left(\lambda^3\mu^2 - \lambda^2\mu^3\right) - \lambda\mu^4 - \frac{\mu^5}{6}\right) \\
& + \frac{1}{7!}\left(\frac{\lambda^5\mu}{12} + \frac{\lambda^4\mu^2}{2} + \frac{23}{24}\lambda^3\mu^3 + \frac{\lambda^2\mu^4}{2} + \frac{\lambda\mu^5}{12}\right) \\
& + \frac{1}{7!}\left(\frac{1}{240}\mu^7 + \frac{1}{30}\lambda\mu^6 + \frac{4}{45}\lambda^2\mu^5 + \frac{1}{15}\lambda^3\mu^4 - \frac{1}{15}\lambda^4\mu^3\right. \\
& \qquad \left. - \frac{4}{45}\lambda^5\mu^2 - \frac{1}{30}\lambda^6\mu - \frac{1}{240}\lambda^7\right) \\
& - \frac{1}{8!}\left(\frac{1}{60}\lambda^7\mu + \frac{2}{15}\lambda^6\mu^2 + \frac{37}{90}\lambda^5\mu^3 + \frac{3}{5}\lambda^4\mu^4 + \frac{37}{90}\lambda^3\mu^5\right. \\
& \qquad \left. + \frac{2}{15}\lambda^2\mu^6 + \frac{1}{60}\lambda\mu^7\right) \\
& + \frac{1}{11!}\left(\frac{\lambda^9}{6} + \frac{25}{3}\lambda^8\mu + 32\lambda^7\mu^2 + 56\lambda^6\mu^3 + 32\left(\lambda^5\mu^4 - \lambda^4\mu^5\right)\right. \\
& \qquad \left. - 56\lambda^3\mu^6 - 32\lambda^2\mu^7 - \frac{25}{3}\lambda\mu^8 - \frac{\mu^9}{6}\right)
\end{aligned}
$$

$$+\frac{\lambda\mu}{11!}\left(\frac{5}{12}(\lambda^8+\mu^8)+\frac{\lambda^7\mu+\lambda\mu^7}{24}+\frac{139}{8}(\lambda^6\mu^2+\lambda^2\mu^6)\right.$$
$$\left.+39(\lambda^5\mu^3+\lambda^3\mu^5)+\frac{305}{6}\lambda^4\mu^4\right).$$

Proposition A.4. *Let L be the Lie algebra freely generated by the symbols P, Q. Under $L \to L/[[L,L],[L,L]]$ the Hausdorff series $H = \log(\exp(P)\cdot\exp(Q))$ maps onto*

$$\begin{aligned}
\overline{H} = {} & (P+Q)+\frac{1}{2}[PQ]+\left(\frac{1}{12}[P^2Q]-\frac{1}{12}[QPQ]\right)-\frac{1}{24}[PQPQ] \\
& +\left(\frac{[Q^3PQ]-[P^4Q]}{720}+\frac{[PQ^2PQ]-[P^2QPQ]}{180}\right) \\
& +\left(\frac{[P^3QPQ]+[PQ^3PQ]}{1440}+\frac{[P^2Q^2PQ]}{360}\right) \\
& +\frac{1}{7!}\left(\frac{1}{6}[P^6Q]+[P^4QPQ]+\frac{4}{3}[P^3Q^2PQ]-\frac{4}{3}[P^2Q^3PQ]\right. \\
& \qquad \left.-[PQ^4PQ]-\frac{1}{6}[Q^5PQ]\right) \\
& -\frac{1}{7!}\left(\frac{1}{12}[P^5QPQ]+\frac{1}{2}[P^4Q^2PQ]+\frac{23}{24}[P^3Q^3PQ]+\frac{1}{2}[P^2Q^4PQ]\right. \\
& \qquad \left.+\frac{1}{12}[PQ^5PQ]\right) \\
& +\frac{1}{7!}\left(\frac{[P^8Q]-[Q^7PQ]}{240}+\frac{[P^6QPQ]-[PQ^6PQ]}{30}\right. \\
& \qquad \left.+\frac{4}{45}([P^5Q^2PQ]-[P^2Q^5PQ])\right) \\
& +\frac{[P^4Q^3PQ]-[P^3Q^4PQ]}{15\cdot 7!} \\
& -\frac{1}{8!}\left(\frac{[P^7QPQ]+[PQ^7PQ]}{60}+\frac{2}{15}([P^6Q^2PQ]+[P^2Q^6PQ])\right) \\
& -\frac{1}{8!}\left(\frac{37}{90}([P^5Q^3PQ]+[P^3Q^5PQ])-\frac{3}{5}[P^4Q^4PQ]\right)+(\textit{higher degree terms}).
\end{aligned}$$

Proof. The coefficient of the term $[P^{n-1}Q^{m-1}PQ]$ in $\overline{H}$ coincides with the coefficient of the term $\lambda^{n-1}\mu^{m-1}$ in $C(\lambda,\mu)$. Hence Proposition A.4 follows from Example A.3. □

Proposition A.5.

(a) *The even part of any compressed Drinfeld associator $\bar{\varphi}(a,b)$ is*

$$\begin{aligned}
Even\big(\bar\varphi(a,b)\big) = {} & \frac{[ab]}{6} - \frac{4[a^3b]+[abab]+4[b^2ab]}{360} + \frac{[a^5b]+[b^4ab]}{945} \\
& + \left(\beta_{31} + \frac{20}{3\cdot 7!}\right)\big(\big[a^3bab\big]+\big[ab^3ab\big]\big) + \left(2\beta_{31} + \frac{13}{7!}\right)\big[a^2b^2ab\big] \\
& - \frac{[a^7b]+[b^6ab]}{9450} \\
& + \left(\frac{\beta_{31}}{6} + \beta_{41} - \frac{1}{4200}\right)\big(\big[a^5bab\big]+\big[ab^5ab\big]\big) \\
& + \left(\frac{2}{3}\beta_{31} + 3\beta_{41} - \frac{113}{45\cdot 7!}\right)\big(\big[a^4b^2ab\big]+\big[a^2b^4ab\big]\big) \\
& + \left(\beta_{31} + 4\beta_{41} - \frac{947}{5\cdot 9!}\right)\big[a^3b^3ab\big] + \frac{[a^9b]+[b^8ab]}{93555} \\
& + (\textit{higher degree terms}).
\end{aligned}$$

(b) *Up to degree* 9 *the odd part of any compressed Drinfeld associator* $\bar\varphi(a,b) \in \widehat{\overline{L}}_3$ *is*

$$\begin{aligned}
Odd\big(\bar\varphi(a,b)\big) = {} & \tilde\beta_{00}\big(\big[a^2b\big]+[bab]\big) + \left(\tilde\beta_{10} + \frac{\tilde\beta_{00}}{6}\right)\big(\big[a^4b\big]+\big[b^3ab\big]\big) \\
& + \left(2\tilde\beta_{10} + \frac{\tilde\beta_{00}}{2}\right)\big(\big[a^2bab\big]+\big[ab^2ab\big]\big) \\
& + \left(\tilde\beta_{20} + \frac{\tilde\beta_{10}}{6} + \frac{\tilde\beta_{00}}{120}\right)\big(\big[a^6b\big]+\big[b^5ab\big]\big) \\
& + \left(3\tilde\beta_{20} + \frac{2}{3}\tilde\beta_{10} + \frac{\tilde\beta_{00}}{24}\right)\big(\big[a^4bab\big]+\big[ab^4ab\big]\big) \\
& + \left(5\tilde\beta_{20} + \frac{7}{6}\tilde\beta_{10} + \frac{\tilde\beta_{00}}{12}\right)\big(\big[a^3b^2ab\big]+\big[a^2b^3ab\big]\big) \\
& + \left(\tilde\beta_{30} + \frac{\tilde\beta_{20}}{6} + \frac{\tilde\beta_{10}}{120} + \frac{\tilde\beta_{00}}{7!}\right)\big(\big[a^8b\big]+\big[b^7ab\big]\big) \\
& + \left(4\tilde\beta_{30} + \frac{5}{6}\tilde\beta_{20} + \frac{\tilde\beta_{10}}{20} + \frac{\tilde\beta_{00}}{720}\right)\big(\big[a^6bab\big]+\big[ab^6ab\big]\big) \\
& + \left(\tilde\beta_{31} + 9\tilde\beta_{30} + 2\tilde\beta_{20} + \frac{2}{15}\tilde\beta_{10} + \frac{\tilde\beta_{00}}{240}\right)\big(\big[a^5b^2ab\big]+\big[a^2b^5ab\big]\big) \\
& + \left(3\tilde\beta_{31} + 13\tilde\beta_{30} + 3\tilde\beta_{20} + \frac{5}{24}\tilde\beta_{10} + \frac{\tilde\beta_{00}}{144}\right)\big(\big[a^4b^3ab\big]+\big[a^3b^4ab\big]\big).
\end{aligned}$$

Proof. It follows from Theorem 1.5(c) by using routine computations. The degree 6 part of the series $\bar{\varphi}^{\mathrm{B}}(a,b)$ from Example 1.4 is obtained from Proposition A.5(a) for $\beta_{31} = -\frac{8}{3\cdot 7!}$. □

Claim A.6.

(a) *The even part of the Drinfeld series* $f^{\mathrm{D}}(\lambda,\mu)$ *from Theorem* 6.2 *starts with*

$$\begin{aligned}
Even\big(f^{\mathrm{D}}(\lambda,\mu)\big) = {} & \frac{1}{6} - \frac{\lambda^2+\mu^2}{90} - \frac{\lambda\mu}{360} + \frac{\lambda^4+\mu^4}{945} + \left(\frac{9}{2}\theta_3^2 + \frac{1}{1260}\right)\big(\lambda^3\mu+\lambda\mu^3\big) \\
& + \left(9\theta_3^2 + \frac{23}{3\cdot 7!}\right)\lambda^2\mu^2 - \frac{\lambda^6+\mu^6}{9450} \\
& + \left(15\theta_3\theta_5 - \frac{2}{3\cdot 7!}\right)\big(\lambda^5\mu+\lambda\mu^5\big) \\
& + \left(45\theta_3\theta_5 + \frac{3}{4}\theta_3^2 - \frac{61}{45\cdot 7!}\right)\big(\lambda^4\mu^2+\lambda^2\mu^4\big) \\
& + \left(60\theta_3\theta_5 + \frac{3}{2}\theta_3^2 - \frac{499}{5\cdot 9!}\right)\lambda^3\mu^3 + \cdots.
\end{aligned}$$

(b) *Up to degree* 7 *the odd part of the Drinfeld series* $f^{\mathrm{D}}(\lambda,\mu)$ *from Theorem* 6.2 *is*

$$\begin{aligned}
Odd\big(f^{\mathrm{D}}(\lambda,\mu)\big) = {} & -3\theta_3(\lambda+\mu) - 5\theta_5\big(\lambda^3+\mu^3\big) - \left(10\theta_5 + \frac{\theta_3}{2}\right)\big(\lambda^2\mu+\lambda\mu^2\big) \\
& - 7\theta_7\big(\lambda^5+\mu^5\big) - \left(21\theta_7 + \frac{5}{6}\theta_5 - \frac{\theta_3}{30}\right)\big(\lambda^4\mu+\lambda\mu^4\big) \\
& - \left(35\theta_7 + \frac{5}{3}\theta_5 - \frac{\theta_3}{24}\right)\big(\lambda^3\mu^2+\lambda^2\mu^3\big) - 9\theta_9\big(\lambda^7+\mu^7\big) \\
& - \left(36\theta_9 + \frac{7}{6}\theta_7 - \frac{\theta_5}{18} + \frac{\theta_3}{315}\right)\big(\lambda^6\mu+\lambda\mu^6\big) \\
& - \left(\frac{9}{2}\theta_3^3 + 84\theta_9 + \frac{7}{2}\theta_7 - \frac{\theta_5}{8} + \frac{\theta_3}{180}\right)\big(\lambda^5\mu^2+\lambda^2\mu^5\big) \\
& - \left(\frac{27}{2}\theta_3^3 + 126\theta_9 + \frac{35}{6}\theta_7 - \frac{7}{36}\theta_5 + \frac{\theta_3}{144}\right)\big(\lambda^4\mu^3+\lambda^3\mu^4\big).
\end{aligned}$$

Proof. Rewrite formula (6.2) in a more explicit form:

$$\begin{aligned}
f^{\mathrm{D}}(\lambda,\mu) = {} & -2\theta_2 - 3\theta_3(\lambda+\mu) - 4\theta_4\big(\lambda^2+\mu^2\big) + \big(2\theta_2^2 - 6\theta_4\big)\lambda\mu - 5\theta_5\big(\lambda^3+\mu^3\big) \\
& + (6\theta_2\theta_3 - 10\theta_5)\big(\lambda^2\mu+\lambda\mu^2\big) - 6\theta_6\big(\lambda^4+\mu^4\big) \\
& + \left(\frac{9}{2}\theta_3^2 + 8\theta_2\theta_4 - 15\theta_6\right)\big(\lambda^3\mu+\lambda\mu^3\big)
\end{aligned}$$

$$+\left(-\frac{4}{3}\theta_2^3+9\theta_3^2+12\theta_2\theta_4-20\theta_6\right)\lambda^2\mu^2-7\theta_7\left(\lambda^5+\mu^5\right)$$
$$+\left(10\theta_2\theta_5+12\theta_3\theta_4-21\theta_7\right)\left(\lambda^4\mu+\lambda\mu^4\right)$$
$$+\left(-6\theta_2^2\theta_3+20\theta_2\theta_5+30\theta_3\theta_4-35\theta_7\right)\left(\lambda^3\mu^2+\lambda^2\mu^3\right)-8\theta_8\left(\lambda^6+\mu^6\right)$$
$$+\left(12\theta_2\theta_6+15\theta_3\theta_5+8\theta_4^2-28\theta_8\right)\left(\lambda^5\mu+\lambda\mu^5\right)$$
$$+\left(30\theta_2\theta_6+45\theta_3\theta_5+24\theta_4^2-9\theta_2\theta_3^2-8\theta_2^2\theta_4-58\theta_8\right)\left(\lambda^4\mu^2+\lambda^2\mu^4\right)$$
$$+\left(\frac{2}{3}\theta_2^4-18\theta_2^2\theta_3-12\theta_2^2\theta_4+40\theta_2\theta_6+60\theta_3\theta_5+34\theta_4^2-70\theta_8\right)\lambda^3\mu^3$$
$$+\left(14\theta_2\theta_7+18\theta_3\theta_6+20\theta_4\theta_5-36\theta_9\right)\left(\lambda^6\mu+\lambda\mu^6\right)-9\theta_9\left(\lambda^7+\mu^7\right)$$
$$+\left(-24\theta_2\theta_3\theta_4-\frac{9}{2}\theta_3^3-10\theta_2^2\theta_5+42\theta_2\theta_7+63\theta_3\theta_6+70\theta_4\theta_5-84\theta_9\right)$$
$$\times\left(\lambda^5\mu^2+\lambda^2\mu^5\right)$$
$$+\left(4\theta_2^3\theta_3-60\theta_2\theta_3\theta_4-\frac{27}{2}\theta_3^3-20\theta_2^2\theta_5+70\theta_2\theta_7+105\theta_3\theta_6\right.$$
$$\left.+120\theta_4\theta_5-126\theta_9\right)\left(\lambda^4\mu^3+\lambda^3\mu^4\right)+\cdots.$$

It remains to substitute θ_{2n} from Example 6.4 and split the even and odd parts. Observe that the Drinfeld series $f^{\mathrm{D}}(\lambda,\mu)$ is obtained from Proposition A.5 for the parameters

$$\beta_{31}=\frac{9}{2}\theta_3^2-\frac{8}{3\cdot 7!},\qquad \beta_{41}=15\theta_3\theta_5-\frac{3}{4}\theta_3^2+\frac{44}{45\cdot 7!},$$
$$\tilde\beta_{00}=-3\theta_3,\qquad \tilde\beta_{10}=-5\theta_5+\frac{\theta_3}{2},\qquad \tilde\beta_{20}=-7\theta_7+\frac{5}{6}\theta_5-\frac{7}{120}\theta_3,$$
$$\tilde\beta_{30}=-9\theta_9+\frac{7}{6}\theta_7-\frac{7}{72}\theta_5+\frac{31}{7!}\theta_3,\qquad \tilde\beta_{31}=-\frac{9}{2}\theta_3^3-3\theta_9+\frac{\theta_3}{630}.$$

The above formulae reprove explicitly Lemma 6.7 and Proposition 6.9 up to degree 7. □

Example A.7. The compressed associators from Corollary 1.6(a), (b) are defined by the series:

(a) $$f^{\mathrm{I}}(\lambda,\mu)=\frac{1}{6}-\frac{4\lambda^2+\lambda\mu+4\mu^2}{360}+\frac{\lambda^4+\mu^4}{945}+\frac{20}{4\cdot 7!}\left(\lambda^3\mu+\lambda\mu^3\right)+\frac{13}{7!}\lambda^2\mu^2$$
$$-\frac{\lambda^6+\mu^6}{9450}-\frac{\lambda^5\mu+\lambda\mu^5}{4200}-\frac{113}{45\cdot 7!}\left(\lambda^4\mu^2+\lambda^2\mu^4\right)-\frac{947}{5\cdot 9!}\lambda^3\mu^3$$
$$+\frac{\lambda^8+\mu^8}{93555}+\cdots;$$

(b) $$f^{\mathrm{II}}(\lambda,\mu)=\frac{1}{6}-\frac{4\lambda^2+\lambda\mu+4\mu^2}{360}+\frac{\lambda^4+\mu^4}{945}-\frac{53}{6\cdot 7!}\big(\lambda^3\mu+\lambda\mu^3\big)-\frac{18}{7!}\lambda^2\mu^2$$
$$-\frac{\lambda^6+\mu^6}{9450}+\frac{\lambda^5\mu+\lambda\mu^5}{11200}-\frac{13}{90\cdot 7!}\big(\lambda^4\mu^2+\lambda^2\mu^4\big)-\frac{431}{5\cdot 9!}\lambda^3\mu^3$$
$$+\frac{\lambda^8+\mu^8}{93555}+\cdots.$$

This follows from Proposition A.5 for $\beta_{31}=\beta_{41}=0$ and $\beta_{31}=-\frac{31}{2\cdot 7!}$, $\beta_{41}=\frac{127}{30\cdot 7!}$.

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