Auto-completion of contours

based on topological persistence

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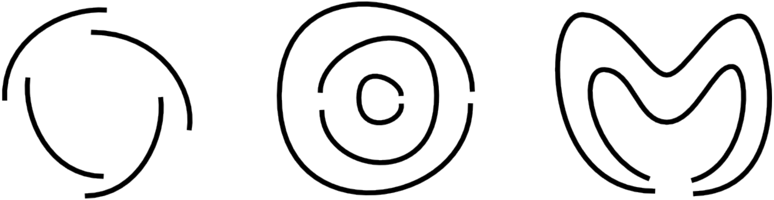
Maps and hand-drawn sketches

Problem: complete all closed contours or paint all regions that they enclose (a segmentation).

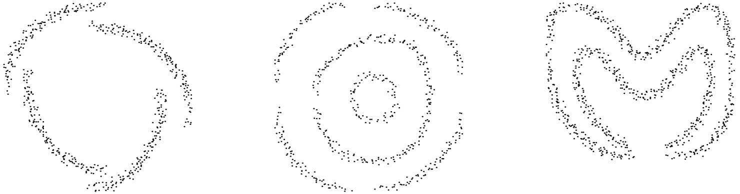


Saund. *Perceptually closed paths*. T-PAMI'03.

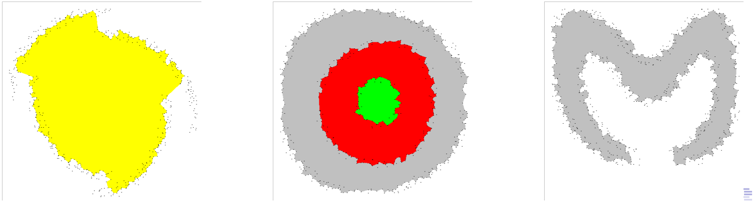
Past algorithms need ad-hoc parameters, e.g. a scale of closeness, weights for quality criteria.

Auto-completion of contours

Input: a dotted 2D image of sparse points without any user-defined parameters, no scale.

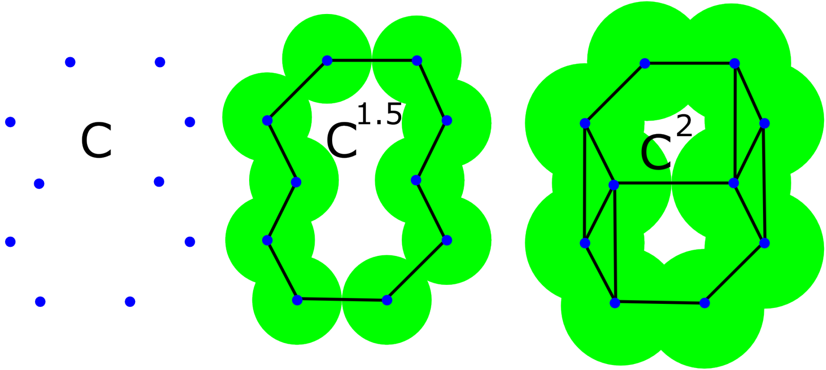


Required output: most ‘persistent’ contours.



From a cloud to some shape

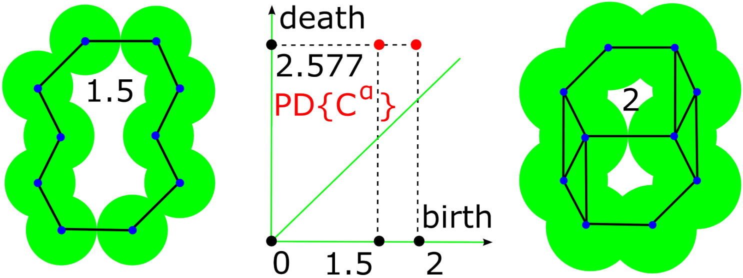
Def : the α-offset of a cloud $C \subset \mathbb{R}^2$ is the union of closed balls $C^\alpha = \cup_{p \in C} B(p; \alpha)$ of a radius α.



Homology group H_1 counts independent cycles: $C^{1.5}$ has 1 cycle, C^2 has 2 cycles, $C^{2.577}$ has 0.

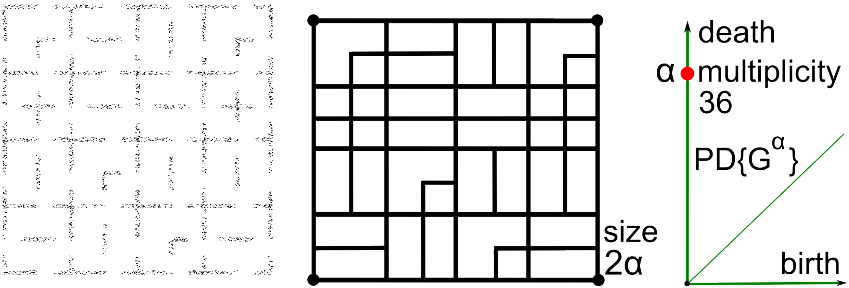
Persistent homology

For any *filtration* $S(\alpha_1) \subset S(\alpha_2) \subset \cdots \subset S(\alpha_m)$, the evolution of homology under linear maps $H_1(S(\alpha_1)) \to \cdots \to H_1(S(\alpha_m))$ is described by pairs $(\mathrm{birth}, \mathrm{death})$ when a class *persists* from its birth $\alpha = \mathrm{birth}$ to its death at $\alpha = \mathrm{death}$.



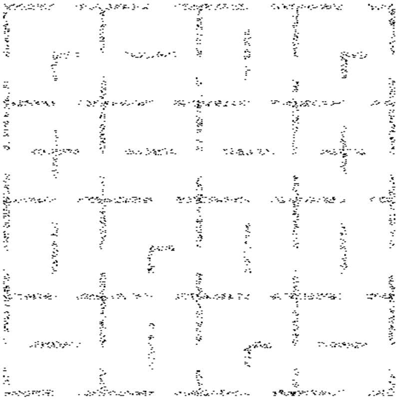
A noisy ε-sample C of a graph

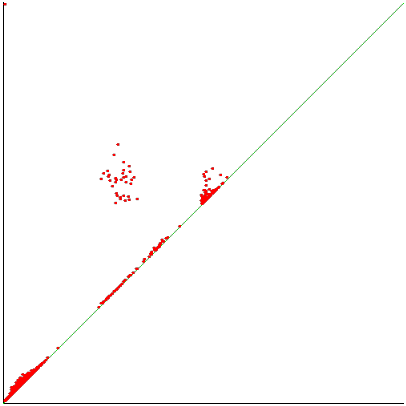
Def: a point cloud C is called a noisy ε-sample of a graph $G \subset \mathbb{R}^2$ if $C \subset G^\varepsilon$ and $G \subset C^\varepsilon$.



Th (Edelsbrunner et al. ’07): any ε-sample C of G has the diagram $\mathrm{PD}\{C^\alpha\}$ ε-close to $\mathrm{PD}\{G^\alpha\}$.

Using stability of persistence

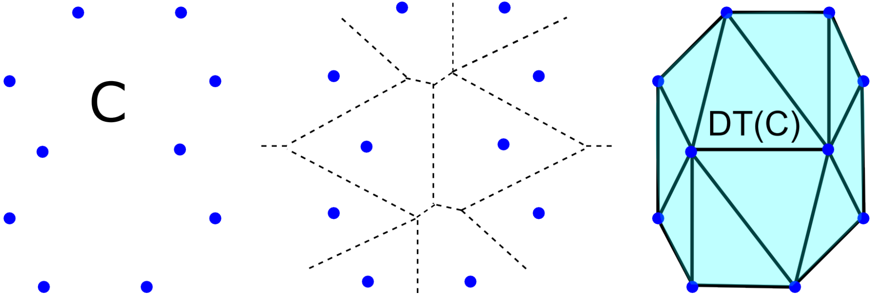




We can find the widest diagonal gap separating 36 points from the rest of persistence diagram.

Delaunay triangulation $\mathrm{DT}(C)$

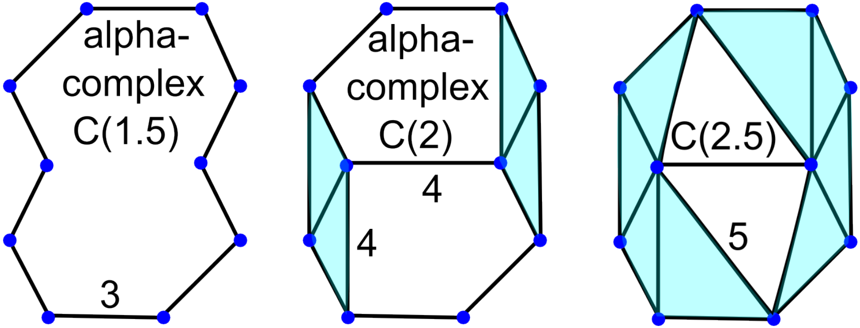
Def: for $C = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$, the Voronoi cell is $V(p_i) = \{q \in \mathbb{R}^2 \,:\, d(q, p_i) \leq d(q, p_j),\, j \neq i\}$.



Def: a Delaunay triangulation $\mathrm{DT}(C)$ is dual to the Voronoi cells and is found in time $O(n \log n)$.

α-complexes $C(\alpha)$ on a cloud C

Def: the α-complex $C(\alpha) \subset \mathrm{DT}(C)$ has edges of length $\leq 2\alpha$ and triangles of circumradius $\leq \alpha$.

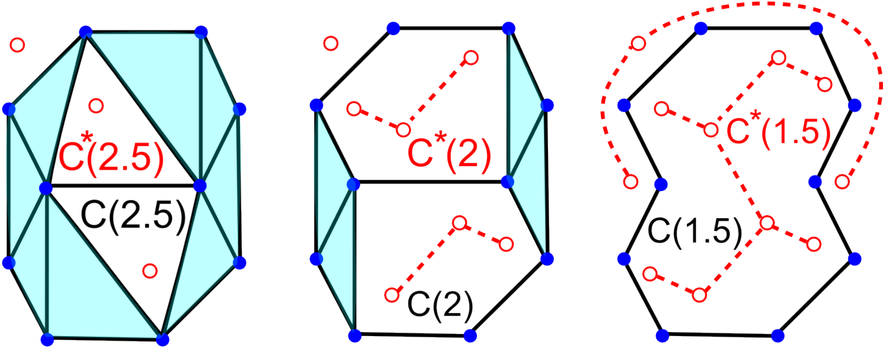


Th (Edelsbrunner '95): any C^{α} deforms to $C(\alpha)$.

Graphs $C^*(\alpha)$ dual to α-complexes

We extend algorithm Attali et al. TopoInVis'09.

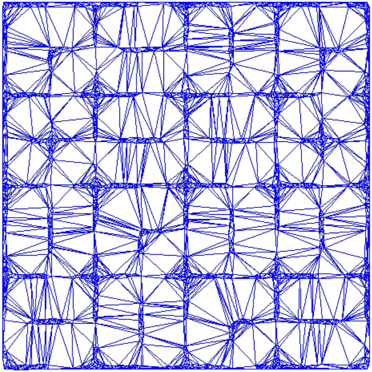
Persistence-sensitive simplification of functions.

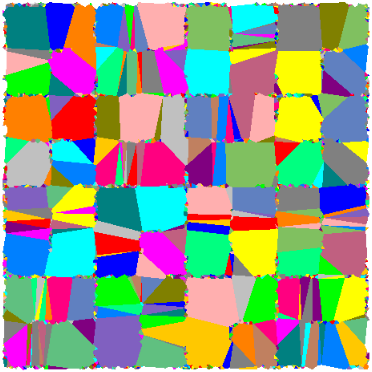


Cycles of $C(\alpha) \subset \mathbb{R}^2$ correspond to connected components of the graph $C^*(\alpha)$ dual to $C(\alpha)$.

Harder than counting cycles

Acute Delaunay triangle is a ‘center of gravity’.

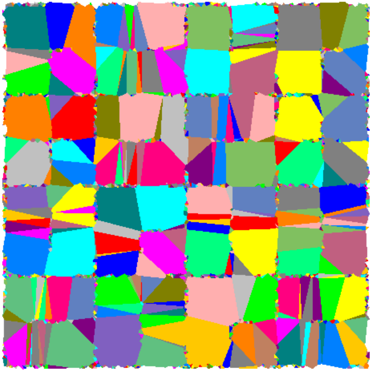


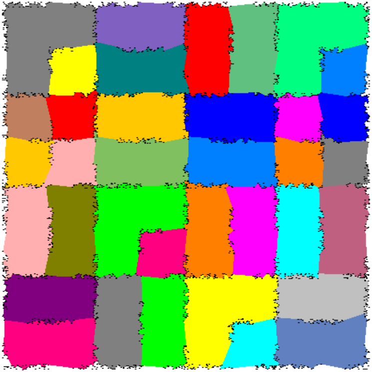


We attach all adjacent non-acute triangles to get an initial segmentation on the right hand side.

Merging initial regions

We maintain adjacency relations when a region merges another one with a higher persistence.

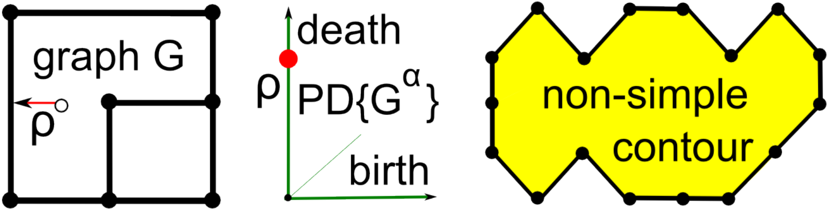




Merger: the older region absorbs the younger.

Simple vs non-simple contours

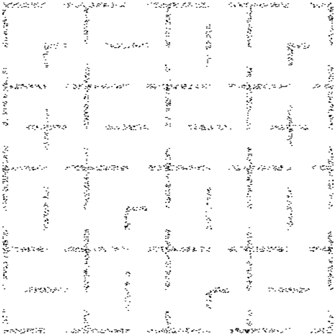
Def: $G \subset \mathbb{R}^2$ is simple if the boundary L of any bounded region in $\mathbb{R}^2 - G$ has a radius $\rho(L)$ such that L^α is circular for $\alpha < \rho(L)$ and $L^\alpha \sim \cdot$ for $\alpha \geq \rho(L)$, so the hole in L^α dies at $\alpha = \rho(L)$.

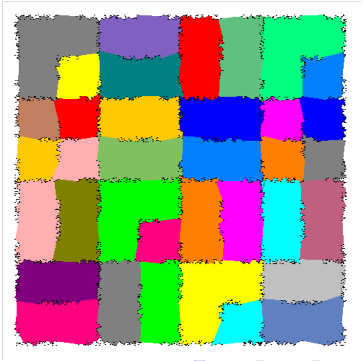


The diagram $\mathrm{PD}\{G^\alpha\}$ of any simple graph has only $(\mathrm{birth}, \mathrm{death}) = (0, \rho)$ in the vertical axis.

Noisy input $\to$ correct output

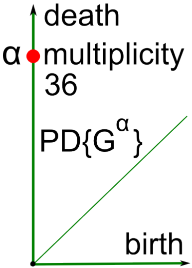
Th (VK'14): let $G \subset \mathbb{R}^2$ be a simple graph with $0 < \rho_1 \leq \cdots \leq \rho_m$ and $\rho_1 > 8\varepsilon + \max\{\rho_{i+1} - \rho_i\}$. For any ε-sample C of G, the algorithm finds m expected contours, they are in the 2ε-offset $G^{2\varepsilon}$.

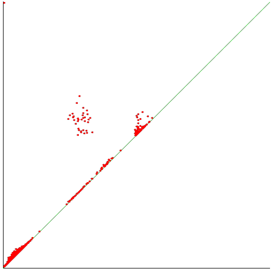


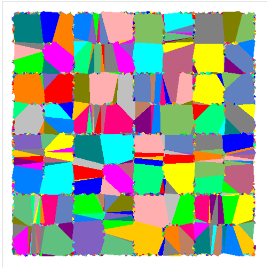


Idea: the widest gap survives

$\rho_1 > 8\varepsilon + \max\{\rho_{i+1} - \rho_i\}$ says that the diagonal gap $\{0 < y - x < \rho_1\}$ is widest under noise.







Also $\varepsilon \geq \max\{\text{birth}\}$ above the widest gap in $\mathrm{PD}\{C^\alpha\}$, hence all edges in a persistent contour $L \subset G$ have half-lengths $\leq \varepsilon$, so $L \subset C^\varepsilon \subset G^{2\varepsilon}$.

Summary and further work

- **input**: 2D point cloud, no extra parameters
- **output**: most persistent closed contours
- **time**: $O(n \log n)$ for any n points in 2D
- 2ε**-approximation** is guaranteed for a noisy ε-sample of a good unknown graph $G \subset \mathbb{R}^2$

- any edge detector: image $\rightarrow$ point cloud, *auto-completion*: cloud $\rightarrow$ object contours
- *extend* to graphs with non-simple contours
- *collaboration* is welcome! kurlin.org/blog