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THREE-PAGE ENCODING AND COMPLEXITY THEORY FOR SPATIAL GRAPHS

V. KURLIN
Department of Mathematical Sciences,
University of Liverpool,
Liverpool L69 7ZL, UK
kurlin@liv.ac.uk

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Dedicated to my father

ABSTRACT

A finitely presented semigroup RSG_n is constructed for $n \geq 2$. The centre of RSG_n encodes uniquely up to rigid ambient isotopy in 3-space all nonoriented spatial graphs with vertices of degree $\leq n$. This encoding is obtained by using three-page embeddings of graphs into the three-page book $T \times I$, where T is the cone on three points, and I is the unit segment. The notion of the three-page complexity for spatial graphs is introduced via three-page embeddings. This complexity satisfies the properties of finiteness and additivity under natural operations.

Keywords: Spatial graph; ambient isotopy; isotopy classification; universal semigroup; three-page embedding; three-page complexity.

Mathematics Subject Classification 2000: 57M25, 57M15, 57M05

1. Introduction

1.1. *Statement of the problem and results*

The method of three-page embeddings is developed for *spatial graphs*, i.e. finite graphs embedded into 3-space. More precisely, the classification problem up to rigid ambient isotopy is reduced to a word problem in finitely presented semigroups. The key idea is to construct suitable three-page embeddings for neighborhoods of vertices in a spatial graph. This construction allows us to reduce the number of generators and defining relations in the universal semigroups for spatial graphs.

1.2. *Motivation of the present research*

The notion of a spatial graph is motivated both theoretically and practically.

Firstly, the classification problem of spatial graphs up to ambient isotopy in $\mathbb{R}^3$ is a special case of the general topological classification of the embeddings into $\mathbb{R}^m$.

The theory of spatial graphs is a natural extension of the classical knot theory to more complicated one-dimensional objects. The isotopy classification problem of spatial graphs was intensively studied in [6, 7]. Many invariants of ordinary links, inlcuding the Alexander polynomial, Jones polynomial and Vassiliev finite-type invariants can be generalized to graphs [11, 15]. The complexity theory for spatial graphs was motivated by Matveev's complexity of 3-dimensional manifolds, which satisfies the properties of finiteness and additivity under connected sum [12].

Secondly, spatial graphs are useful mathematical models for long protein molecules in molecular biology. For instance, it is of importance whether the shape of a molecule is symmetric under reflection [14].

1.3. *Review of the previous results*

An embedding of a link in a structure resembling an open book with finitely many pages was probably for the first time considered by Brunn in 1898 [1]. More exactly, Brunn proved that each link is isotopic to a link that can be projected to the plane with only one singular point. Later studies of such embeddings gave a new link invariant, *the arc index* [3]. It turned out that each link can be embedded into *the book with three pages* $\mathbb{Y} = T \times I$, where T is the cone on 3 points and $I \approx [0, 1]$.

In 1999 Dynnikov encoded all nonoriented links in $\mathbb{R}^3$ by *three-page diagrams* and reduced the isotopy classification to a word problem in a finitely presented semigroup [4, 5]. To be more precise these diagrams will be called *three-page embeddings*, see the formal definition in Sec. 3.1. Formally, Dynnikov constructed the semigroup DS such that its centre encodes all nonoriented links up to ambient isotopy in $\mathbb{R}^3$. Vershinin and the author have extended the three-page approach to spatial *3-graphs* (graphs with vertices of degree only 3) [8] and to *singular knots* (links with finitely many double intersections in general position) [10].

1.4. *Basic definitions*

A finite 1-dimensional CW-complex G is called a *finite graph*. Every 0-dimensional (respectively, 1-dimensional) cell of G is said to be *a vertex* (respectively, *an edge*) of the graph G. Since *hanging edges* having an endpoint of degree 1 cannot be knotted, they are excluded. All graphs are considered up to homeomorphism.

Definition 1.1 (k-vertices of a graph, n-graphs and J-graphs).

(a) A vertex $A \in G$ is called *a k-vertex* (i.e. A has *degree* k), if the graph G has exactly k edges attached to A.

(b) Fix an integer $n \geq 2$. If a graph G has k-vertices for $k = 2, \ldots, n$ only, then G is said to be *an n-graph*.

(c) Let $J = \{j_1, \ldots, j_k\}$ be any set of integers $j_l \geq 3$. If a graph G has k-vertices, where either $k = 2$ or $k \in J$, then G will be called *a J-graph*.

We consider only nonoriented graphs, possibly disconnected. Self-loops and multiple edges are allowed. For instance, a 2-graph is the disjoint union of circles.

Definition 1.2 (a spatial graph, rigid and non-rigid isotopies). Let G be a nonoriented finite graph. We work in the PL-category, i.e. the images of the edges of G under an embedding in $\mathbb{R}^3$ are finite polygonal lines.

(a) *A spatial* (or *knotted*) graph is a subset $G \subset \mathbb{R}^3$, homeomorphic to G. We also assume that a neighborhood of any vertex of G lies in a plane.
(b) *An ambient PL-isotopy* between spatial graphs $G, H \subset \mathbb{R}^3$ is a continuous family of PL-homeomorphisms $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,1]$, such that $\phi_0 = \mathrm{id}$, $\phi_1(G) = H$.
(c) If in addition, at each moment $t \in [0,1]$ of the isotopy ϕ_t, a neighborhood of every vertex of the graph $\phi_t(G)$ lies in a plane depending on t, then ϕ_t is called *a rigid isotopy*. Otherwise ϕ_t is said to be *a non-rigid isotopy*.

For example, *singular knots* are spatial $\{4\}$-graphs considered up to rigid isotopy [10]. See a singular knot and a 3-graph in Figs. 11, 12, respectively (Sec. 3). For spatial 3-graphs, any non-rigid isotopy can be transformed into a rigid one. We may keep three arcs at each 3-vertex of a graph in a (non-constant) plane. For spatial n-graphs with $n > 3$, a non-rigid isotopy can permute edges at any vertex. Except subsection 5.4 spatial n-graphs are considered up to rigid isotopy only.

Definition 1.3 (the encoding alphabet $\mathbb{A}_n$). For each $n \geq 2$, let us consider the following *encoding alphabet*:

$$\mathbb{A}_n = \{a_i,\ b_i,\ c_i,\ d_i,\ x_{m,i} \mid i \in \mathbb{Z}_3,\ 3 \leq m \leq n\}.$$

The index i always belongs to the group $\mathbb{Z}_3 = \{0,1,2\}$. In particular, for $n = 2$, we get the Dynnikov alphabet from [4]:

$$\mathbb{A}_2 = \{a_0,\ a_1,\ a_2,\ b_0,\ b_1,\ b_2,\ c_0,\ c_1,\ c_2,\ d_0,\ d_1,\ d_2\}.$$

The alphabet $\mathbb{A}_n$ contains exactly $3(n+2)$ letters.

Definition 1.4 (the universal semigroups RSG_n and NSG_n).

(a) Let RSG_n be the semigroup generated by the letters of $\mathbb{A}_n$ and relations (1.1)–(1.10). Everywhere the integer parameters m, p, q will satisfy the inequalities

$$3 \leq m \leq n, \quad 2 \leq p \leq \frac{n+1}{2}, \quad 2 \leq q \leq \frac{n}{2}.$$

$$d_0 d_1 d_2 = 1; \tag{1.1}$$

$$b_i d_i = d_i b_i = 1; \tag{1.2}$$

$$a_i = a_{i+1} d_{i-1}, \quad b_i = a_{i-1} c_{i+1}, \quad c_i = b_{i-1} c_{i+1}, \quad d_i = a_{i+1} c_{i-1}; \tag{1.3}$$

$$x_{2p-1,i-1} = d_{i-1}^{p-1}(x_{2p-1,i} d_{i+1}) b_{i-1}^{p-2}, \quad x_{2q,i-1} = d_{i-1}^{q-2}(b_{i+1} x_{2q,i} d_{i+1}) b_{i-1}^{q-2}; \tag{1.4}$$

$$x_{2p-1,i}d_i^{p-1} = a_i(x_{2p-1,i}d_i^{p-1})c_i, \quad b_i^{p-1}x_{2p-1,i}b_i = a_i(b_i^{p-1}x_{2p-1,i}b_i)c_i; \quad (1.5)$$

$$d_i x_{2q,i}d_i^{q-1} = a_i(d_i x_{2q,i}d_i^{q-1})c_i, \quad b_i^{q-1}x_{2q,i}b_i = a_i(b_i^{q-1}x_{2q,i}b_i)c_i; \quad (1.6)$$

$$(d_i c_i)w = w(d_i c_i), \quad \text{where } w \in \{c_{i+1},\ b_i d_{i+1} d_i,\ x_{m,i+1}\}; \quad (1.7)$$

$$uv = vu, \quad \text{where} \begin{array}{l} u \in \{a_i b_i,\ b_{i-1}d_i d_{i-1}b_i,\ x_{2p-1,i}b_i,\ d_i x_{2q,i}b_i\}, \\ v \in \{a_{i+1},\ b_{i+1},\ c_{i+1},\ b_i d_{i+1} d_i,\ x_{m,i+1}\}; \end{array} \quad (1.8)$$

$$(x_{2p-1,i}b_i)D_{p,i} = D_{p-1,i}(x_{2p-1,i}b_i), \quad \text{where } D_{k,i} = d_i^k d_{i+1}^k d_{i-1}^k (k \geq 1); \quad (1.9)$$

$$(d_i x_{2q,i} b_i)D_{q,i} = D_{q,i}(d_i x_{2q,i} b_i). \quad (1.10)$$

(b) Let us introduce the semigroup NSG_n generated by the letters of $\mathbb{A}_n$ and relations (1.1)–(1.8), (1.9′) $x_{m,i}b_i(d_i^2 d_{i+1}^2 d_{i-1}^2) = x_{m,i}b_i$, $3 \leq m \leq n$, $i \in \mathbb{Z}_3$.

(c) For any set $J = \{j_1, \ldots, j_k\}$ of integers $j_l \geq 3$, denote by RSG_J the semigroup generated by the letters $\{a_i, b_i, c_i, d_i, x_{m,i} \mid i \in \mathbb{Z}_3,\ m \in J\}$ and those relations (1.1)–(1.10) that contain only these letters. Let the semigroup NSG_J be generated by the same letters and relations (1.1)–(1.8), (1.9′) for $m \in J$.

The semigroups RSG_n and NSG_n are monoids, the empty word $\emptyset$ is the identity element. The generators b_i and d_i are mutually inverse by (1.2). The generators a_i and c_i have no inverses in RSG_n and NSG_n. A geometric interpretation for the letters of $\mathbb{A}_n$ and relations (1.1)–(1.10) will be given in Sec. 2. One of the relations in (1.2) is superfluous: it can be obtained from (1.1) and the other relations in (1.2). Then the total number of relations (1.1)–(1.10) is $3(n^2 + 7n - 2)$.

The semigroups $RSG_2 = NSG_2$ generated by 12 letters a_i, b_i, c_i, d_i $(i \in \mathbb{Z}_3)$ and 48 defining relations (1.1)–(1.3), (1.7)–(1.8), that contain only the letters of $\mathbb{A}_2$, coincide with the Dynnikov semigroup DS from [5]. The semigroups $RSG_3 \cong NSG_3$ and $RSG_{\{4\}} \not\cong NSG_{\{4\}}$ are generated by 15 letters and 84 relations. Denote by $|J|$ the number of elements of a set $J = \{j_1, \ldots, j_k\}$, $j_l \geq 3$. The semigroups RSG_J and NSG_J are generated by $3(4 + |J|)$ letters and $3(16 + 11|J| + |J|^2)$ relations.

Definition 1.5 (the automorphisms ρ_n, ε_n and mirror image of a graph).

(a) Consider the following map on the letters of $\mathbb{A}_n$: $\rho(a_i) = c_i$, $\rho(b_i) = d_i$,

$$\rho(c_i) = a_i, \quad \rho(d_i) = b_i, \quad \rho(x_{2p-1,i}) = x_{2p-1,i}b_i c_i, \quad \rho(x_{2q,i}) = x_{2q,i}.$$

By the formula $\rho(uv) = \rho(v)\rho(u)$ the map ρ extends to

the involutive automorphisms $\rho_n : RSG_n \to RSG_n$ and $\varepsilon_n : NSG_n \to NSG_n$.

Similarly, define the morphisms $\rho_J : RSG_J \to RSG_J$ and $\varepsilon_J : NSG_J \to NSG_J$.

(b) *The mirror image* of a spatial graph $G \subset \mathbb{R}^3$ is the spatial graph $\bar{G} \subset \mathbb{R}^3$ reflection symmetric to G.

In Corollaries 1.9 and 1.11 below we shall consider the problem to decide whether a spatial graph G is isotopic to its mirror image $\bar{G}$.

1.5. *Main results*

By Theorems 1.6 and 1.7 the isotopy classification of nonoriented spatial graphs reduces to a pure algebraic word problem in finitely presented semigroups.

Theorem 1.6.

(a) *A spatial n-graph G is encoded by an element $w_G \in RSG_n$.*
(b) *Two spatial n-graphs $G, H \subset \mathbb{R}^3$ are* rigidly isotopic *in $\mathbb{R}^3$ if and only if the corresponding elements of the semigroup RSG_n are equal*: $w_G = w_H$.
(c) *An element $w \in RSG_n$ encodes a spatial n-graph if and only if the element w is central*, *i.e. w commutes with any element of RSG_n. Moreover*, *there is an algorithm to decide whether a given element $w \in RSG_n$ is central. The algorithm is linear in the length of the word w.*

Theorem 1.6 means that the centre of RSG_n encodes uniquely all spatial n-graphs up to rigid isotopy in $\mathbb{R}^3$. Proposition 5.4 in Sec. 5.2 shows that the whole semigroup RSG_n describes a wider class of *rigid three-page tangles*.

Theorem 1.7. *The centre of the semigroup NSG_n encodes all spatial n-graphs up to* non-rigid isotopy *in $\mathbb{R}^3$. There is an algorithm to decide whether a given element $v \in NSG_n$ is central. The algorithm is linear in the length of the word v.*

Theorem 1.8. *Let $\{G\}$ be the set of all nonoriented spatial graphs considered up to homeomorphism $f : S^3 \to S^3$, which can reverse the orientation of S^3. There exists* a complexity function $\text{tp} : \{G\} \to \mathbb{N}$ *with the following properties*:

(a) *for any $k \in \mathbb{N}$, there is a finite number of spatial graphs G with* $\text{tp}(G) = k$;
(b) *there are well-defined operations on spatial graphs*: the disjoint union $G \sqcup H$, a vertex sum $G * H$, an edge sum $G \vee H$ *such that*

$$\text{tp}(G \sqcup H) = \text{tp}(G * H) = \text{tp}(G) + \text{tp}(H) + 2 \quad \textit{and} \quad \text{tp}(G \vee H) = \text{tp}(G) + \text{tp}(H) + 3.$$

Theorem 1.8 was motivated by Matveev's complexity for 3-manifolds [12].

Theorems 1.6 and 1.7 imply several algebraic and geometric corollaries.

Corollary 1.9.

(a) *Let a spatial graph G be encoded by $w_G \in RSG_n$. The graph G is* rigidly isotopic *to its mirror image if and only if $\rho_n(w_G) = w_G$ in RSG_n.*
(b) *Let a spatial n-graph $G \subset \mathbb{R}^3$ be encoded by $v_G \in NSG_n$. The graph G is* non-rigidly *isotopic to its mirror image $\bar{G}$ if and only if $\varepsilon_n(v_G) = v_G$ in NSG_n.*

An element w of a semigroup S is *invertible*, if w has left and right inverses.

Corollary 1.10.

(a) *When $2 \le k < n$ the natural inclusion $RSG_k \to RSG_n$ is a monomorphism of semigroups. The group of the invertible elements of RSG_n coincides with the*

Dynnikov group $DG \subset RSG_2$, generated by 2 letters:

$$DG = \langle x, y \mid \big[[x,y], x^2yx^{-2}\big] = \big[[x,y], y^2xy^{-2}\big] = \big[[x,y], [x^{-1}, y^{-1}]\big] = 1\rangle,$$

where $[x,y] = xyx^{-1}y^{-1}$.

(b) *When* $2 \le k < n$ *the natural inclusion* $NSG_k \to NSG_n$ *is a monomorphism of semigroups. The group of the invertible elements of* NSG_n *coincides with* DG.

The commutator subgroup of DG is the braid group B_∞ on infinitely many strings [5]. The method of three-page embeddings can be applied to J-graphs.

Corollary 1.11.

(a) *The centre of the semigroup* RSG_J (*respectively*, NSG_J) *encodes all spatial* J*-graphs up to rigid* (*respectively, non-rigid*) *isotopy in* $\mathbb{R}^3$. *There is an algorithm to decide whether a given element* $w \in RSG_J$ (*respectively*, $v \in NSG_J$) *is central. The algorithm is linear in the length of the given word.*

(b) *Let a spatial graph* G *be encoded by* $w_G \in RSG_J$ (*respectively, by* $v_G \in NSG_J$). *The graph* G *is* rigidly (*respectively*, non-rigidly) *isotopic to its mirror image if and only if* $\rho_J(w_G) = w_G$ *in* RSG_n (*respectively*, $\varepsilon_J(v_G) = v_G$ *in* NSG_J).

(c) *For any subset* $K \subset J$, *the natural inclusions* $RSG_K \to RSG_J$ *and* $NSG_K \to NSG_J$ *are monomorphisms of semigroups. The groups of the invertible elements of the semigroups* RSG_J *and* NSG_J *coincide with the Dynnikov group* DG.

The following corollary extends Brunn's result on embeddings of links [1].

Corollary 1.12. *Any spatial* J*-graph* $G \subset \mathbb{R}^3$ *is non-rigidly isotopic to a spatial graph that can be projected to* $\mathbb{R}^2$ *with only one singular point.*

It is well-known that not any finite graph can be topologically embedded into $\mathbb{R}^2$. What minimal space contains all finite graphs? Theorem 1.6(a) implies

Corollary 1.13. *Any finite graph* (*possibly with hanging edges*) *can be topologically embedded into* $T \times I$, *where* T *is the cone on three points*, $I \approx [0,1]$.

1.6. *Outline of the paper*

In Sec. 2, the generators and relations (1.1)–(1.10) of the semigroup RSG_n are described in a clear geometric way. Section 2.4 contains a scheme for the proof of Theorem 1.6. In Sec. 3, the proofs for Theorem 1.6(a) and Corollary 1.13 are given. Theorems 1.6(b), 1.6(c), 1.7 and Corollaries 1.9–1.12 are proved in Sec. 5.

The hard part of Theorem 1.6(b) is a particular case of Proposition 5.4 proved in Sec. 5.2. In Sec. 6, we deduce Lemma 5.5 used in the proof of Proposition 5.4. Section 7 discusses various approaches to the classification of spatial graphs via three-page embeddings. Theorem 1.8 is proved in Sec. 7.3.

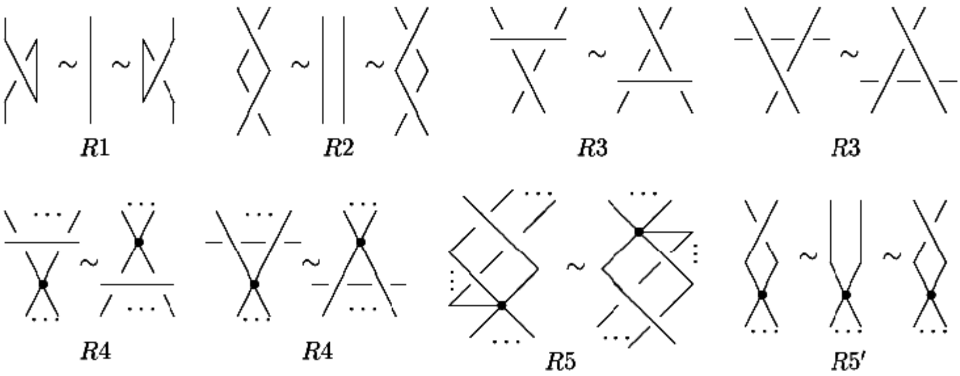

Fig. 1. The Reidemeister moves for spatial graphs in $\mathbb{R}^3$.

2. Geometric Interpretation of Semigroups RSG_n and NSG_n

2.1. *Reidemeister moves for spatial graphs*

We start with an analogue of classical Reidemeister theorem for spatial graphs.

Theorem 2.1 [6]. *Any spatial graph can be represented by its plane diagram up to generalized Reidemeister moves* $R1$–$R5$ *in Fig.* 1. *In the case of non-rigid isotopy, the move* $R5'$ *is taken instead of* $R5$.

Subdivisions of edges and edge breaks are omitted. In Fig. 1 *dots* between two arcs denote finitely many arcs. Reidemeister moves are local, but two-dimensional. Theorem 1.6(b) states that moves (1.1)–(1.10) on words in $\mathbb{A}_n$ also generate any rigid isotopy of graphs. Moves (1.1)–(1.10) are local and 1-dimensional.

2.2. *Geometric interpretation of the alphabet* $\mathbb{A}_n$

The alphabet $\mathbb{A}_n$ was introduced in Definition 1.3. Here the letters of $\mathbb{A}_n$ will be associated to particular geometric patterns of three-page embeddings.

Definition 2.2 (the three-page book $\mathbb{Y}$, the axis α, the pages P_i).

(a) *The three-page book* is the product $\mathbb{Y} = T \times I$, where T is the cone on three points and $I \approx [0, 1]$ is the oriented segment.
(b) The interval $I - \partial I$ is homeomorphic to the line $\mathbb{R}$ and is said to be *the axis* α. The set $\mathbb{Y} - \partial\mathbb{Y}$ is the union $P_0 \cup P_1 \cup P_2$ of three half-planes with common oriented boundary $\partial P_0 = \partial P_1 = \partial P_2 = \alpha$. The half-planes P_i are called *the pages* of $\mathbb{Y}$.

In Figs. 2 and 3, every letter of $\mathbb{A}_n$ encodes a local embedding into the book $\mathbb{Y}$. In these figures, the page P_0 lies above the axis α, the pages P_1, P_2 are below α, and P_2 is below P_1, i.e. the arcs in P_2 are shown in dashed lines.

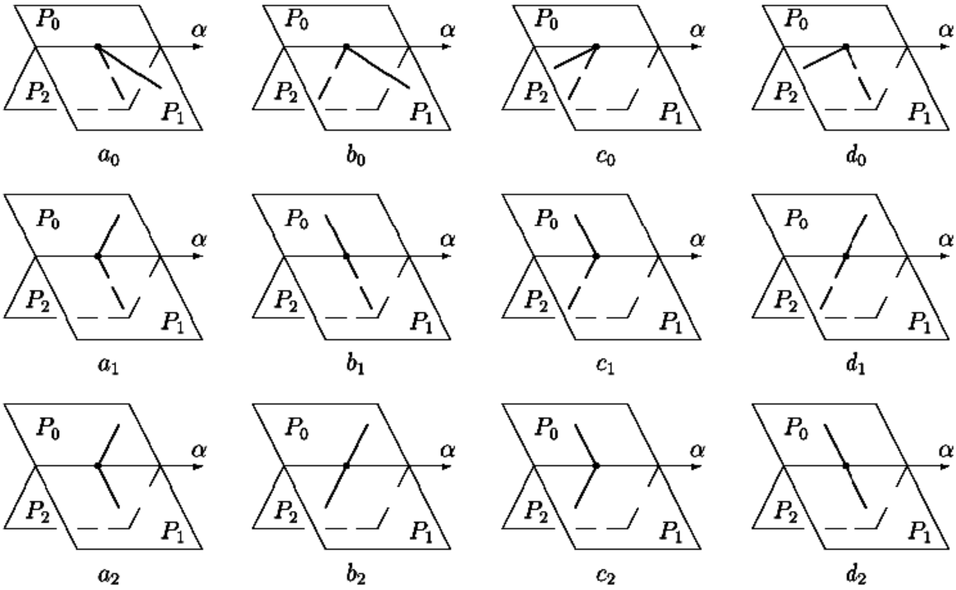

Fig. 2. The Dynnikov letters of the alphabet $\mathbb{A}_2$.

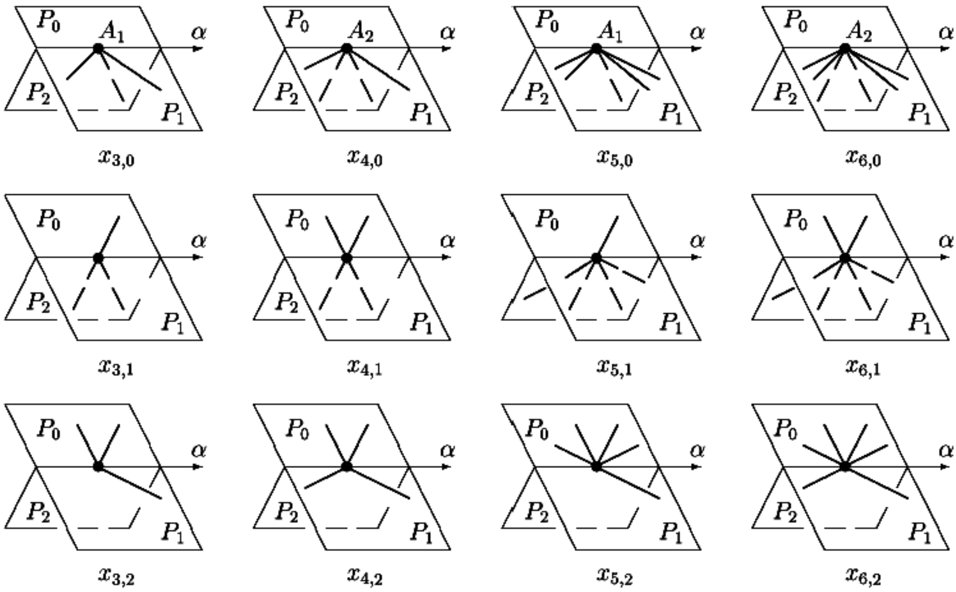

Fig. 3. The letters for vertices of degrees 3, 4, 5, 6.

In Fig. 2, every letter with the index $i \in \mathbb{Z}_3 = \{0, 1, 2\}$ denotes an embedding of two arcs into the bowed disk $P_{i-1} \cup P_{i+1}$. In Fig. 3, each letter $x_{m,i}$ encodes an embedding of a neighborhood of an m-vertex into the same disk $P_{i-1} \cup P_{i+1}$.

More exactly, one of the arcs at each $(2p-1)$-vertex A_1 lies in P_{i-1} and points toward to the positive direction of the axis α. The other arcs at A_1 lies in P_{i+1}. Exactly $p-1$ of these arcs point toward to the positive direction of α, and another ones point toward to the negative direction of α.

Similarly, two arcs at every $2q$-vertex A_2 lie in P_{i-1}, one of them points toward to the positive direction of α, and the other one points toward to the negative direction of α. Also the other $2q-2$ arcs at A_2 lie in P_{i+1}, exactly $q-1$ of them point toward to the positive direction of α, and the other ones point toward to the negative direction of α.

The disk $P_{i-1} \cup P_{i+1}$ does not lie in a plane. Without loss of generality one can assume that during a rigid isotopy a neighborhood of every vertex lies in such a bowed disk. Attaching one local picture of Fig. 2 or 3 to another according to the direction of α, we get a three-page embedding representing a given word w in $\mathbb{A}_n$. The words a_0c_0, a_1c_1, a_2c_2 encode *the unknot*, i.e. a circle embedded into $\mathbb{R}^2$.

2.3. *Local isotopy moves in the three-page approach*

Relations (1.1)–(1.10) can be performed by rigid isotopies denoted by $\sim$. During rigid isotopies in Figs. 4–10 neighborhoods of vertices are in bowed disks.

2.4. *Scheme for the proof of Theorem 1.6*

The formal definition of a three-page embedding of a spatial graph is given in Sec. 3.1. In Sec. 3.2, a three-page embedding $G \subset \mathbb{Y}$ of a spatial graph $G \subset \mathbb{R}^3$ will be construsted from a plane diagram of G.

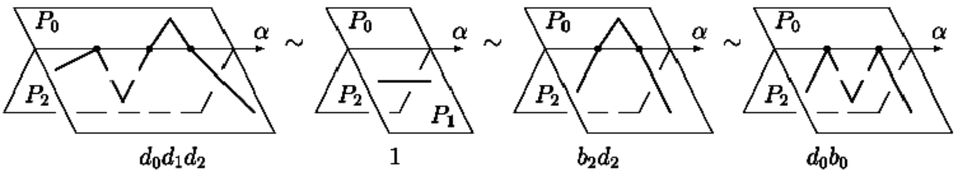

Fig. 4. Relations (1)–(2) between invertible elements.

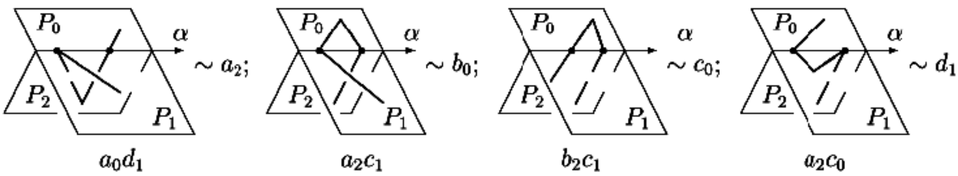

Fig. 5. Relations (3) are trivial moves at intersection points.

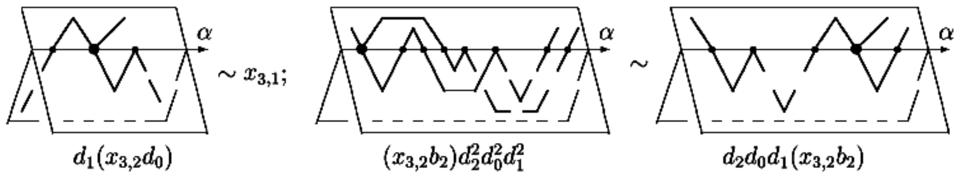

Fig. 6. Relations (4_p) and (9) are twistings of arcs at a $(2p-1)$-vertex.

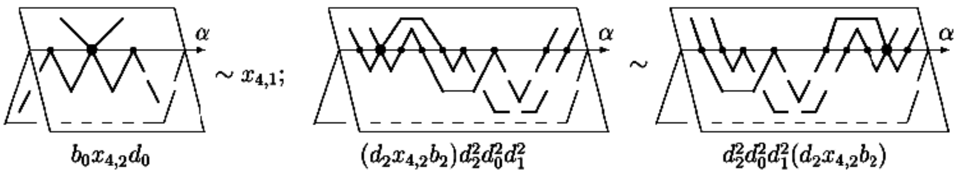

Fig. 7. Relations (4_q) and (10) are twistings of arcs at a $2q$-vertex.

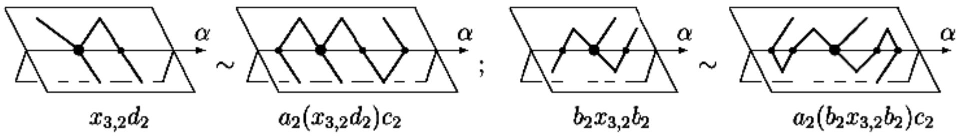

Fig. 8. Relations (5) are rotatings of arcs at a $(2p-1)$-vertex.

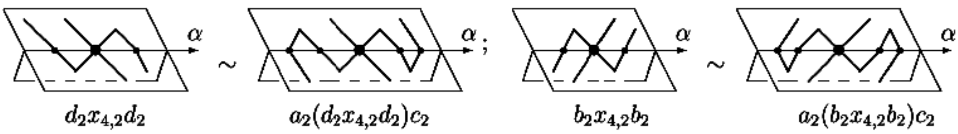

Fig. 9. Relations (6) are rotatings of arcs at a $2q$-vertex.

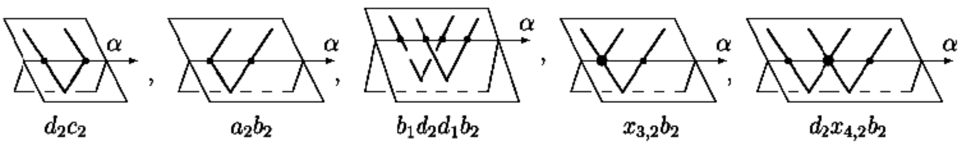

Fig. 10. These elements commute with $a_0, b_0, c_0, b_2d_0d_2, x_{m,0}$ in (1.7)–(1.8).

Theorem 1.6(a) will be proved in Sec. 3.3, where the constructed three-page embedding $G \subset \mathbb{Y}$ is encoded by a word in the alphabet $\mathbb{A}_n$.

The hard part of Theorem 1.6(b) is that any rigid isotopy of spatial n-graphs decomposes into *elementary moves* (1.1)–(1.10). For the proof of Theorem 1.6(b), the notions of graph tangles and three-page tangles are introduced in Sec. 4. The semigroup RGT_n of all rigid graph tangles will be described by generators and defining relations (4.1)–(4.13), see Lemma 4.4 in Sec. 4.2.

Every three-page embedding of a graph can be transformed to an almost balanced tangle, which is a particular three-page tangle. The semigroup RBT_n of almost balanced tangles is turned out to be isomorphic to RGT_n, see Lemma 5.3 in Sec. 5.1. The isomorphism $\varphi : RGT_n \to RBT_n$ maps relations (4.1)–(4.13) to relations φ(4.1)–φ(4.13) between words in the alphabet $\mathbb{A}_n$.

Any three-page embedding of a graph can be represented by a graph tangle of RGT_n. By Lemma 4.4 any isotopy between three-page embeddings of graphs decomposes into moves (4.1)–(4.13) between graph tangles, and hence into *simple isotopies* φ(4.1)–φ(4.13) between almost balanced tangles.

So, it remains to deduce relations φ(4.1)–φ(4.13) of RBT_n from relations (1.1)–(1.10) of RSG_n. This is Lemma 5.5, which will be checked in Sec. 6. Theorem 1.6 (c) is proved in Sec. 5.3 by using knot-like three-page tangles.

3. Three-Page Embeddings of Spatial Graphs

3.1. *The definition of a three-page embedding*

Let G be a finite graph, $A \in G$ be its point. Any small segment $\gamma \subset G$ with endpoint $A \in G$ is *an arc* of G. There are exactly k arcs at each k-vertex of G.

Definition 3.1 (a three-page embedding of a spatial graph). Suppose that a spatial graph $G \subset \mathbb{R}^3$ is contained in the three-page book $\mathbb{Y} \subset \mathbb{R}^3$. The embedding $G \subset \mathbb{Y}$ is called *a three-page embedding*, if

(a) all vertices of the graph G lie in the axis α, see Fig. 11;
(b) the intersection $G \cap \alpha = A_1 \cup \cdots \cup A_k$ is a non-empty finite set of points;
(c) the arcs at each 2-vertex $A_l \in G \cap \alpha$ lie in different pages P_i, P_j $(i \neq j)$;
(d) *balance:* neighborhoods of vertices in G look like pictures of Fig. 3.

Since the arcs lying in a page P_i are not intersected, then by isotopy inside $\mathbb{Y}$ we may secure the following condition, which will be always assumed:

(e) *monotonicity*: for each $i \in \mathbb{Z}_3$, the restriction of the orthogonal projection $\mathbb{Y} \to \alpha$ to each connected component of the intersection $G \cap P_i$ is a monotonic function.

3.2. *Construction of a three-page embedding from a plane diagram*

Let D be *a plane diagram* of a spatial graph $G \subset \mathbb{R}^3$. Formally, $D \subset \mathbb{R}^2$ is a plane graph with vertices of two types: ones correspond to initial vertices of G and the others represent usual *crossings* in a planar projection of the spatial graph G.

Definition 3.2 (bridges, upper and lower arcs of a plane diagram). Let us choose following *bridges* and *arcs* in a plane diagram D.

(a) For each crossing of the plane diagram D, let us mark a small arc (*a regular bridge*) in the overcrossing arc. See the left pictures of Figs. 11 and 12.

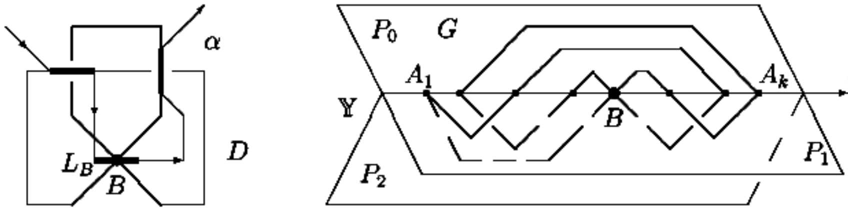

Fig. 11. The spatial graph G is encoded by the word $w_G = a_0a_1b_2d_1x_{4,1}d_2c_1c_2$.

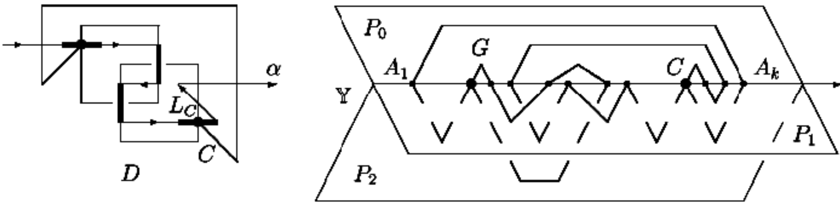

Fig. 12. The spatial graph G is encoded by $w_G = a_1x_{3,1}d_2a_1b_2b_0c_1d_0x_{3,1}d_2b_2c_1$.

(b) For every $2q$-vertex $B \in G$, let us call two neighboring arcs at B *upper arcs* and call the other $2q-2$ arcs at B *lower arcs*. Mark a small segment (*a singular bridge* L_B) containing B such that the upper arcs are separated from the lower arcs by the bridge L_B in a neighborhood of B. See the left picture of Fig. 11.

(c) For every $(2p-1)$-vertex $C \in G$, let us call one arc at C *an upper arc* and call the other $2p-2$ arcs at C *lower arcs*. Then mark a small segment (*a singular bridge* L_C) containing C such that the upper arc is separated from the lower arcs by the bridge L_C in a neighborhood of C. See the left picture of Fig. 12.

In the plane of D choose a non-self-intersecting oriented path α with the following properties.

(i) The ends of the path α lie far from the diagram D.
(ii) The path α goes through each bridge only once.
(iii) The path α intersects transversally the diagram D outside the bridges.
(iv) For every vertex $A \in G$, the upper arcs at A lie to the left of α. The lower arcs at A lie to the right of α. See the left pictures of Figs. 11 and 12.
(v) For every $2q$-vertex $B \in G$, an upper arc and exactly $q-1$ lower arcs at B meet the oriented path α to the left of B. Similarly, another upper arc and the other $q-1$ lower arcs at B meet α to the right of B. See the left picture of Fig. 11.
(vi) The upper arc and $p-1$ lower arcs at each $(2p-1)$-vertex C meet α to the right of C. The other $p-1$ arcs at C meet α to the left of C. See Fig. 12.

Such a path α can be found as follows. Let us consider the bridges of D, i.e. finitely many arcs in the plane of D. Draw an arbitrary path α through the bridges according to conditions (i), (ii) and (iv). Condition (iii) will hold, if the path α is made to be in general position with respect to the diagram D.

Assume that condition (vi) does not hold for a $(2p-1)$-vertex C. For instance, let the upper arc γ at C meet α to the left of C. Slightly perturb the path α to the right of C by a move similar to $R2$ in Fig. 1 in such a way that the upper arc meets α to the right of C, see the left puicture of Fig. 12. We deal with condition (v) similarly. Deform D in such a way that α becomes a line segment and

(vii) the restriction of the orthogonal projection $\mathbb{R}^2 \to \alpha \approx \mathbb{R}$ to each connected component of $D - \alpha$ is a monotone function.

Denote by P_0 (respectively, P_2) the upper part (respectively, the lower part) of $\mathbb{R}^2 - \alpha$, see the right pictures of Figs. 11 and 12. Let us attach the third page P_1 at the axis α and push out the regular bridges into P_1 in such a way that each regular bridge becomes *a trivial arc* (two line segments).

Conditions (a)–(c) of Definition 3.1 hold by the construction. Conditions (iv)–(vii) of this subsection imply condition (d).

3.3. *Encoding of a three-page embeddings*

Here we encode a three-page embedding and simultaneously prove Theorem 1.6(a).

Proof of Theorem 1.6(a). Take a plane diagram D of a given spatial graph G. Starting with D construct a three-page embedding $G \subset \mathbb{Y}$ as in Sec. 3.2.

An arbitrary three-page embedding is uniquely determined by its small part near the axis α. Indeed, in order to reconstruct the whole embedding it suffices to join all opposite-directed arcs in each page starting with innermost arcs.

Due to conditions (c)–(e) of Definition 3.1, only the patterns in Figs. 2 and 3 may occur in a three-page embedding near the axis α. Denote by W_n the set of all words in the alphabet $\mathbb{A}_n$ including the empty word $\emptyset$. For the three-page embedding $G \subset \mathbb{Y}$, let us write one by one the letters of $\mathbb{A}_n$ corresponding to the intersection points of $G \cap \alpha$. One gets a word $w_G \in W_n$, see Figs. 11 and 12.

Finally, consider the word w_G as an element of RSG_n. Note that one can rotate any three-page embedding around the axis α. Then each spatial graph G is represented by at least three words obtained from w_G by the index map $i \mapsto i+1$. □

Proof of Corollary 1.13. Draw a given graph G (possibly with hanging edges) in $\mathbb{R}^2$ in such a way that its edges are intersected in double points only. Near each double point push exactly one arc out of $\mathbb{R}^2$. For the obtained spatial graph $G \subset \mathbb{R}^3$, let us construct a three-page embedding $G \subset \mathbb{Y}$ as in Sec. 3.2. □

3.4. *Balanced words in the alphabet* $\mathbb{A}_n$

The encoding procedure from Sec. 3.2 cannot give all words of W_n.

Briefly, a word $w \in W_n$ is called *balanced*, if it encodes a three-page embedding of a spatial graph. There exists the following geometric criterion for a balanced word: "in each page P_i all the arcs have to be joined to one another". Arcs in an unbalanced three-page embedding can recede to infinity without meeting one another. One can restate this criterion algebraically in terms of the alphabet $\mathbb{A}_n$.

Definition 3.3 (balanced bracket expressions).

(a) An expression containing left and right round brackets is *a bracket expression*.
(b) A bracket expression β is called *balanced*, if (by reading β from left to right) in each place the number of the left brackets is not less than the number of the right ones, and their total numbers are equal.

Definition 3.4 (balanced words in the alphabet $\mathbb{A}_n$).

(a) Let us consider the following substitution

$$a_i,\, b_i,\, c_i,\, d_i,\, x_{m,i} \to \varnothing,\ a_{i\pm 1},\, b_{i-1},\, d_{i+1},\, x_{2p-1,i-1} \to (;\ b_{i+1},\, c_{i\pm 1},\, d_{i-1} \to);$$

$$x_{2q,i+1} \to)(;\ x_{2q,i-1} \to)^{q-1}(^{q-1};\ x_{2p-1,i+1} \to)^{p-1}(^{p-1}, \text{ where } (^j = j \text{ brackets } (.$$

As usual, we have $i \in \mathbb{Z}_3$, $3 \le m \le n$, $2 \le p \le \frac{n+1}{2}$ and $2 \le q \le \frac{n}{2}$. Denote by $\beta_i(w)$ the resulting expression after the above substitution into a word $w \in W_n$.
(b) A word w is called *i-balanced*, if the bracket expression $\beta_i(w)$ is balanced.
(c) A word w is said to be *balanced*, if it is i-balanced for each $i \in \mathbb{Z}_3$.

So, a word w is balanced if and only if all three bracket expressions $\beta_i(w)$ are balanced. For example, the word w in Fig. 11 is balanced and provides

$$\beta_0(w) = ((()())), \quad \beta_1(w) = ()(), \quad \beta_2(w) = (())().$$

Denote by $W_{n,i}$ the set of all i-balanced words in $\mathbb{A}_n$. Definition 3.4 implies that there is an algorithm to decide whether a word w is balanced. The algorithm computes the differences of the left and right brackets by reading $\beta_i(w)$ from left to right and has a linear complexity in the length of w.

The intersection $W_{n,0} \cap W_{n,1} \cap W_{n,2}$ is the set of all balanced words in $\mathbb{A}_n$. Lemma 5.7 will show that this set encodes the centres of RSG_n and NSG_n.

4. Graph Tangles and Three-Page Tangles

4.1. *Graph tangles*

The category of tangles without vertices was studied by Turaev [16]. Let us take two horizontal half-lines given by coordinates: $(r, 0, 0)$ and $(r, 0, 1)$, $r \in \mathbb{R}_+$. For all $k \in \mathbb{N}$, let us mark the points $(k, 0, 0)$, $(k, 0, 1)$ on these half-lines.

Definition 4.1 (rigid graph tangles). Let Γ be a nonoriented disconnected infinite graph with vertices of degree $\le n$. *A graph tangle* is a subset of the

3-dimensional layer $\{0 \le z \le 1\}$, homeomorphic to Γ, such that the following conditions hold (see Fig. 13)

(a) the set of the 1-vertices of Γ coincides with the set of the marked points $\{(k,0,0),(k,0,1) \mid k \in \mathbb{N}\}$;
(b) the connected components of Γ lying sufficiently far from the origin $0 \in \mathbb{R}^3$ are the line segments joining points $(k,0,0)$ and $(j,0,1)$ such that the difference $k-j$ is constant for all large j;
(c) a neighborhood of each vertex $A \in \Gamma$ lies in a plane or in a bowed disk.

If an isotopy of graph tangles inside $\{0 < z < 1\}$ keeps condition (c), then the corresponding isotopy classes of graph tangles are *rigid graph tangles.*

Graph tangles are represented by their plane diagrams similarly to spatial graphs. *The product* $\Gamma_1 \times \Gamma_2$ of two graph tangles is the graph tangle obtained by attaching the top half-line of Γ_2 to the bottom half-line of Γ_1 and then by contracting the new layer $\{0 \le z \le 2\}$ to the initial one.

The rigid isotopy classes of graph tangles form a semigroup RGT_n. *The unit graph tangle* $1 \in RGT_n$ consists of the vertical line segments joining the points $(k,0,0)$ and $(k,0,1)$, $k \in \mathbb{N}$. Let us consider the rigid graph tangles in Fig. 13:

$$\mathbb{T}_n = \{\xi_k,\ \eta_k,\ \sigma_k,\ \sigma_k^{-1},\ \lambda_{m,k} \mid k \ge 1,\ 3 \le m \le n\}.$$

For any $k \in \mathbb{N}$, the tangle $\eta_k \xi_k$ is the unknot added to the unit $1 \in RGT_n$.

4.2. *The semigroup RGT_n of rigid graph tangles*

We work in the PL-category, i.e. graph tangles consist of finite polygonal lines.

Definition 4.2. (the graph Γ_{xz}, the extremal points and peculiarities of Γ_{xz}).

(a) Denote by Γ_{xz} the image of a graph tangle $\Gamma \subset \{0 \le z \le 1\}$ under the projection to the xz-plane, see Fig. 13.
(b) *The extremal points* of Γ_{xz} are the images under the xz-projection of local maxima and minima of the z-coordinate on the interiors of the edges of Γ.
(c) The images on the xz-plane of the vertices of Γ (except the 1-vertices), the crossings and extremal points of Γ_{xz} are called *the peculiarities* of Γ_{xz}.

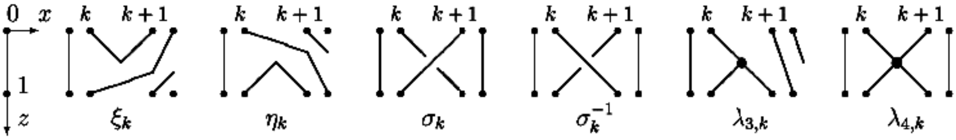

Fig. 13. The generators for the graph tangles.

Each tangle in Fig. 13 has exactly one peculiarity. The tangles ξ_k and η_k have an extremal point, σ_k and σ_k^{-1} have a crossing, $\lambda_{m,k}$ contains an m-vertex.

Definition 4.3 (graphs $\boldsymbol{\Gamma_{xz}}$ in general position). Let us say that a graph Γ_{xz} is in *general position* on the xz-plane, if the following conditions hold (see Fig. 13):

(a) the graph Γ_{xz} has finitely many peculiarities;
(b) each crossing is not an extremal point;
(c) for every $(2p-1)$-vertex $C \in \Gamma_{xz}$, exactly $p-1$ arcs at endpoint C *go up*, i.e. these $p-1$ arcs point toward to the negative direction of the z-axis, the other p arcs at endpoint C go down;
(d) for each $2q$-vertex $B \in \Gamma_{xz}$, exactly q arcs at endpoint B go up, the other q arcs at enpoint B go down;
(e) no two peculiarities lie in *a horizontal line* parallel to the x-axis.

Lemma 4.4 extends Turaev's results [16] to graph tangles.

Lemma 4.4. *The semigroup RGT_n is generated by the letters of $\mathbb{T}_{\boldsymbol{n}}$ and defining relations* (4.1)–(4.13), *where* $k \geq 1$, $3 \leq m \leq n$, $2 \leq p \leq \frac{n+1}{2}$, $2 \leq q \leq \frac{n}{2}$.

$$\xi_k\xi_l = \xi_{l+2}\xi_k,\ \xi_k\eta_l = \eta_{l+2}\xi_k,\ \xi_k\sigma_l = \sigma_{l+2}\xi_k,\ \xi_k\lambda_{m,l} = \lambda_{m,l+2}\xi_k \quad s(l \geq k); \tag{4.1}$$

$$\eta_k\xi_l = \xi_{l-2}\eta_k,\ \eta_k\eta_l = \eta_{l-2}\eta_k,\ \eta_k\sigma_l = \sigma_{l-2}\eta_k,\ \eta_k\lambda_{m,l} = \lambda_{m,l-2}\eta_k \quad (l \geq k+2); \tag{4.2}$$

$$\sigma_k\xi_l = \xi_l\sigma_k, \quad \sigma_k\eta_l = \eta_l\sigma_k, \quad \sigma_k\sigma_l = \sigma_l\sigma_k, \quad \sigma_k\lambda_{m,l} = \lambda_{m,l}\sigma_k \quad (l \geq k+2); \tag{4.3}$$

$$\left.\begin{array}{lll}
\lambda_{2p-1,k}\xi_l = \xi_{l-1}\lambda_{2p-1,k}, & \lambda_{2p-1,k}\sigma_l = \sigma_{l-1}\lambda_{2p-1,k}, & \\
\lambda_{2p-1,k}\eta_l = \eta_{l-1}\lambda_{2p-1,k}, & \lambda_{2p-1,k}\lambda_{m,l} = \lambda_{m,l-1}\lambda_{2p-1,k} & (l \geq k+p), \\
\lambda_{2q,k}\xi_l = \xi_l\lambda_{2q,k}, & \lambda_{2q,k}\sigma_l = \sigma_l\lambda_{2q,k}, & \\
\lambda_{2q,k}\eta_l = \eta_l\lambda_{2q,k}, & \lambda_{2q,k}\lambda_{m,l} = \lambda_{m,l}\lambda_{2q,k} & (l \geq k+q);
\end{array}\right\} \tag{4.4}$$

$$\eta_{k+1}\xi_k = 1 = \eta_k\xi_{k+1}; \tag{4.5}$$

$$\eta_{k+2}\sigma_{k+1}\xi_k = \sigma_k^{-1} = \eta_k\sigma_{k+1}\xi_{k+2}; \tag{4.6}$$

$$\eta_{k+p-1}\lambda_{2p-1,k+1}\xi_k = \lambda_{2p-1,k} = \eta_k\lambda_{2p-1,k+1}\xi_{k+p}; \tag{4.7}$$

$$\eta_{k+q}\lambda_{2q,k+1}\xi_k = \lambda_{2q,k} = \eta_k\lambda_{2q,k+1}\xi_{k+q}; \tag{4.8}$$

$$\eta_k\sigma_k = \eta_k, \quad \sigma_k\xi_k = \xi_k; \tag{4.9}$$

$$\sigma_k\sigma_k^{-1} = 1 = \sigma_k^{-1}\sigma_k; \tag{4.10}$$

$$\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}; \tag{4.11}$$

$$\left.\begin{array}{ll}
\lambda_{2p-1,k+1}\Sigma_{k,p} = \Sigma_{k,p-1}\lambda_{2p-1,k}, & \lambda_{2p-1,k}\bar{\Sigma}_{k,p} = \bar{\Sigma}_{k,p-1}\lambda_{2p-1,k+1}, \\
\lambda_{2q,k+1}\Sigma_{k,q} = \Sigma_{k,q}\lambda_{2q,k}, & \lambda_{2q,k}\bar{\Sigma}_{k,q} = \bar{\Sigma}_{k,q}\lambda_{2q,k+1}, \\
\textit{where } \Sigma_{k,l} = \sigma_k\sigma_{k+1}\cdots\sigma_{k+l-1}, & \bar{\Sigma}_{k,l} = \sigma_{k+l-1}\cdots\sigma_{k+1}\sigma_k \ (l \geq 1);
\end{array}\right\} \tag{4.12}$$

$$\left.\begin{array}{l}\lambda_{2p-1,k}\Sigma'_{k,p-1} = \Sigma'_{k,p-2}\lambda_{2p-1,k}, \quad \lambda_{2q,k}\Sigma'_{k,q-1} = \Sigma'_{k,q-1}\lambda_{2q,k}, \ \textit{where} \\ \Sigma'_{k,0} = 1, \ \Sigma'_{k,l} = \sigma^{-1}_{k+l-1}(\sigma^{-1}_{k+l-2}\sigma^{-1}_{k+l-1})\cdots(\sigma^{-1}_{k}\sigma^{-1}_{k+1}\cdots\sigma^{-1}_{k+l-1}), \ l \geq 1.\end{array}\right\} \quad (4.13)$$

Proof. Let $\Gamma \subset \{0 \leq z \leq 1\}$ be a graph tangle. The graph Γ_{xz} can be moved to general position on the xz-plane by a slight deformation. Then the xz-plane splits by horizontal lines into strips each of that contains exactly one peculiarity of Γ_{xz}.

Consider the peculiarities of Γ_{xz} in succession from top to bottom one by one. Write the corresponding generators in Fig. 13 from left to right. We get a word u_Γ in the alphabet $\mathbb{T}_n$. The generators ξ_k, η_k represent extremal points; σ_k, σ_k^{-1} denote overcrossings and undercrossings; the letter $\lambda_{m,k}$ corresponds to an m-vertex.

It suffices to prove that any rigid isotopy of graph tangles decomposes into *elementary isotopies* (4.1)–(4.13). By [6, Theorem 2.1] and consideration relating to the general position an arbitrary rigid isotopy of graph tangles decomposes into the following moves:

(i) an isotopy in the class of diagrams in general position;
(ii) an isotopy interchanging the vertical positons of two peculiarities;
(iii) creation or annihilation of a pair of neighbouring extremal points;
(iv) an isotopy of a crossing or of a vertex near an extremal point;
(v) Reidemeister moves $R1 - R5$ in Fig. 1.

The type (i) isotopies preserve the constructed word u_Γ in $\mathbb{T}_n$.

The type (ii) isotopies are described by relations (4.1) – (4.4).
The type (iii) isotopies correspond to relations (4.5).

In [16, proof of Lemma 3.4], it was shown that all isotopies of a crossing near an extremal point decompose geometrically into relations (4.6). Similarly, one can check that relations (4.7)–(4.8) are sufficient to realize the type (iv) isotopies.

Reidemeister moves $R1$–$R5$ correspond to relations (4.9)–(4.13), respectively. □

4.3. *Three-page tangles*

Let us consider three half-lines that have a common endpoint in the horizontal plane $\{z = 0\}$. For example, let

$$Y = \{x \geq 0, y = z = 0\} \cup \{y \geq 0, x = z = 0\} \cup \{x \leq 0, y = z = 0\} \subset \{z = 0\}.$$

Mark the integer points on the half-lines:

$$\{(j, 0, 0), (0, k, 0), (-l, 0, 0) \mid j, k, l \in \mathbb{N}\}.$$

Let $I \subset \mathbb{R}^3$ be the line segment joining $(0, 0, 0)$ and $(0, 0, 1)$. Put (see Fig. 14):

$$P_0 = \{x > 0, y = z = 0\} \times I, \quad P_1 = \{y > 0, x = z = 0\} \times I,$$

$$P_2 = \{x > 0, y = z = 0\} \times I.$$

Then $Y \times I$ is the three-page book with the pages P_i, see Definition 2.2.

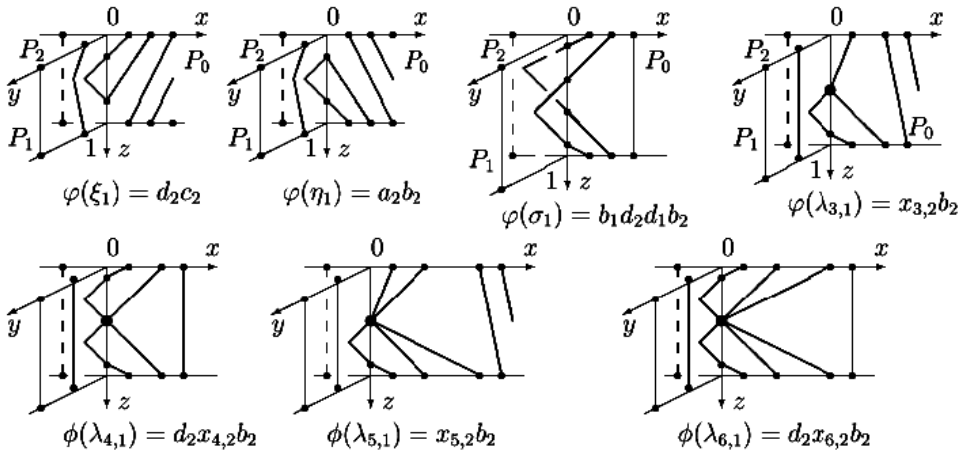

Fig. 14. The three-page tangles associated to the graph tangles from $\mathbb{T}_6$.

Definition 4.5 (rigid three-page tangles). Let Γ be a nonoriented disconnected infinite graph with vertices of degree $\leq n$. *A graph tangle* is a subset $Y \times I$, homeomorphic to Γ, such that

(a) the set of the 1-vertices of Γ coincides with the set of the marked points

$$\{(j,0,0),(j,0,1),(0,k,0),(0,k,1),(-l,0,0),(-l,0,1) \mid j,k,l \in \mathbb{N}\};$$

(b) all vertices of degree ≥ 3 lie in the segment I, see Fig. 14;
(c) *finiteness*: the intersection $\Gamma \cap I = A_1 \cup \cdots \cup A_m$ is a finite set of points;
(d) the arcs at each 2-vertex $A_j \in \Gamma \cap I$ lie in different pages P_i, P_j $(i \neq j)$;
(e) *balance:* neighborhoods of vertices $A_j \in \Gamma \cap \alpha$ look like patterns in Fig. 3;
(f) *monotonicitiy*: for any $i \in \mathbb{Z}_3$, the restriction of the orthogonal projection $Y \times I \to I$ to each connected component of $\Gamma \cap P_i$ is a monotone function;
(g) for each $i \in \mathbb{Z}_3$, all connected components of Γ lying in the page P_i sufficiently far from the origin $0 \in \mathbb{R}^3$ are parallel line segments.

If three-page tangles are considered up to rigid (respectively, non-rigid) isotopy in the layer $\{0 < z < 1\}$, then they are called *rigid* (respectively, *non-rigid*). □

The reader may compare Definition 4.5 with Definitions 2.2 and 4.1.

The rigid isotopy classes of three-page tangles with vertices of degree $\leq n$ form a semigroup. Proposition 5.4 will show that this semigroup is isomorphic to RSG_n. Any three-page tangle $\Gamma \subset \{0 \leq z \leq 1\}$ can be encoded by a word w_Γ in the alphabet $\mathbb{A}_n$ (see Figs. 2 and 3) in the same way as in Sec. 2.3.

5. Proof of Main Results

Theorems 1.6(b) and 1.6(c) will be proved in Secs. 5.2 and 5.3, respectively. Corollaries 1.9(a) and 1.10(a) will be checked at the end of Sec. 5.3. Section 5.4 is devoted to the proofs of Corollaries 1.9(b), 1.10(b), 1.11 and 1.12.

5.1. *The semigroup RBT_n of almost balanced tangles*

We are going to select almost balanced tangles among three-page tangles. The semigroup RBT_n of rigid almost balanced tangles will turn out to be isomorphic to the semigroup RGT_n of rigid graph tangles.

Definition 5.1 (almost balanced tangles up to rigid isotopy).

(a) A three-page tangle $\Gamma \subset Y \times I$ is called *almost balanced*, if the corresponding word w_Γ in $\mathbb{A}_n$ is simultaneously 1-balanced and 2-balanced. Equivalently, one can assume that the marked points lying in P_1, P_2 are joined in an *almost balanced tangle* by vertical line segments parallel to the z-axis.

(b) An isotopy of a tangle inside the layer $\{0 < z < 1\}$ is said to be *rigid*, if a neighborhood of each vertex lies in a plane or in a bowed disk. Denote by RBT_n the semigroup of the rigid isotopy classes of almost balanced tangles.

Any graph tangle in the sense of Definition 4.1 can be embedded into $Y \times I$ in such a way that its 1-vertices lie in the half-lines $\{x \geq 0, y = z = 0\}$ and $\{x \geq 0, y = 0, z = 1\}$. Then we may add two infinite families of vertical line segments in P_1, P_2 and get an almost balanced tangle.

Since graph tangles and three-page tangles are defined up to rigid isotopy in the layer $\{0 < z < 1\}$, then there is a non-canonical homomorphism $RGT_n \to RBT_n$.

Definition 5.2 (the isomorphism $\varphi : RGT_n \to RBT_n$ of semigroups). Let us take the map $\varphi : RGT_n \to RBT_n$ defined on the generators of the semigroup RGT_n as follows ($k \in \mathbb{N}$, see Fig. 14):

$$\left.\begin{array}{lll} \varphi(\xi_k) = d_2^k c_2 b_2^{k-1}, & \varphi(\sigma_k) = d_2^{k-1} b_1 d_2 d_1 b_2^k, & \varphi(\lambda_{2p-1,k}) = d_2^{k-1} x_{2p-1,2} b_2^k, \\ \varphi(\eta_k) = d_2^{k-1} a_2 b_2^k, & \varphi(\sigma_k^{-1}) = d_2^k b_1 b_2 d_1 b_2^{k-1}, & \varphi(\lambda_{2q,k}) = d_2^k x_{2q,2} b_2^k. \end{array}\right\} \tag{5.1}$$

Lemma 5.3. *The map φ is a well-defined isomorphism of semigroups.*

Proof. Let us show that rigidly isotopic graph tangles go to rigidly isotopic three-page tangles under the map φ. By Definitions 4.5 and 5.1 tangles in the semigroups RGT_n and RBT_n are considered up to rigid isotopy in $\{0 < z < 1\}$. Therefore, the map φ is a well-defined monomorphism of semigroups.

Let us construct the inverse map $\psi : RBT_n \to RGT_n$. To each almost balanced tangle $\Gamma \in RBT_n$ associate the graph tangle $\psi(\Gamma) \in RGT_n$ given by the diagram constructed as follows. According to the property of being almost balanced, we assume that all line segments of Γ lying in the pages P_1, P_2 are vertical.

Deleting these vertical line segments from Γ we obtain a graph tangle $\psi(\Gamma)$ in the sense of Definition 4.1. The composition $\psi \circ \varphi : RGT_n \to RGT_n$ is identical on the generators of RGT_n, see Fig. 14. So, φ and ψ are mutually inverse. □

Denote by $\varphi(4.1) - \varphi(4.13)$ the relations between words in the alphabet $\mathbb{A}_n$ which are obtained from relations (4.1)–(4.13) of the semigroup RGT_n under the isomorphism $\varphi : RGT_n \to RBT_n$ defined by formulae (5.1).

5.2. *The hard part of Theorem 1.6(b)*

If two words $u, v \in W_n$ represent the same element of the semigroup RSG_n, then call them *equivalent* and denote by $u \sim v$. Theorem 1.6(b) is a particular case of Proposition 5.4 below. Actually, any spatial graph can be represented by a three-page tangle encoded by a balanced word. If two balanced three-page tangles are rigidly isotopic, then the corresponding words are equal by Proposition 5.4.

Proposition 5.4. *The semigroup of the rigid isotopy classes of three-page tangles is isomorphic to the semigroup RSG_n.*

Proof. As was mentioned at the end of Sec. 4.3, to each three-page tangle one can associated a word w in the alphabet $\mathbb{A}_n$ and hence an element of RSG_n. Conversely, each element $w \in RSG_n$ can be completed to form a three-page tangle by adding three families of parallel line segments on each page P_i, $i \in \mathbb{Z}_3$.

Indeed, let us draw local three-page embeddings (see Figs. 2 and 3) representing the letters of the word w. Then extend all arcs until some of them meet one another and the other arcs come to the boundary of $Y \times I$.

For every page P_i, let us consider the arcs coming to the half-lines $P_i \cap \{z = 0\}$ and $P_i \cap \{z = 1\}$. We may assume that these arcs end the points marked by $1, 2, \ldots, n_i$ (say) and $1, 2, \ldots, m_i$, respectively. In the page P_i, consider the points marked by $n_i + k$ and $m_i + k$ for all $k \in \mathbb{N}$. Join the points marked by $n_i + k$ and $m_i + k$ by adding infinitely many parallel line segments lying in the page P_i. We get a three-page tangle $\Gamma(w)$ in the sense of Definition 4.5. For example, Fig. 14 shows the three-page tangles corresponding to the following elements of RSG_n:

$$d_2c_2, \quad a_2b_2, \quad b_1d_2d_1b_2, \quad x_{3,2}b_2, \quad d_2x_{4,2}b_2, \quad x_{5,2}b_2, \quad d_2x_{6,2}b_2.$$

Relations (1.1)–(1.10) of the semigroup RSG_n can be realized easily by rigid isotopies inside $\{0 < z < 1\}$, see Figs. 4–10. It remains to prove that any rigid isotopy of three-page tangles can be decomposed into *the elementary isotopies* corresponding to (1.1)–(1.10). It suffices to do this for almost balanced tangles.

Actually, take a three-page tangle Γ associated to a word w_Γ. Consider the marked points lying in $P_1 \cap \{z = 0\}$ and $P_1 \cap \{z = 1\}$ that are joined in Γ with points in I. Let n_1 and m_1 be the maximal indices of the above points lying in

$P_1 \cap \{z = 0\}$ and $P_1 \cap \{z = 1\}$, respectively. Let n_2 and m_2 be the maximal indices of the similar marked points lying in $P_2 \cap \{z = 0\}$ and $P_2 \cap \{z = 1\}$, respectively. The letters a_0 and b_0 have $n_1 = n_2 = 0, m_1 = m_2 = 1$ and $n_1 = m_2 = 0, m_1 = n_2 = 1$, respectively. The numbers n_i, m_i defined here are the same as above.

For all almost balanced tangle, we have $n_1 = m_1 = n_2 = m_2 = 0$. The word $b_1^{n_2} d_2^{n_1} w b_2^{m_1} d_1^{m_2}$ is 1-balanced and 2-balanced. Due to the invertibility of the generators b_i and d_i such a transformation send equivalent words to equivalent ones.

By Lemma 5.3 to each almost balanced tangle of RBT_n one can associate a graph tangle in the sense of Definition 4.1. For such tangles, any rigid isotopy is already decomposed into relations $\varphi(4.1)$–$\varphi(4.13)$ in Lemma 4.4. Then Proposition 5.4 follows from Lemma 5.5, which will be checked in Sec. 6.3. □

Lemma 5.5. *Relations* $\varphi(4.1)$*–*$\varphi(4.13)$ *follow from relations* (1.1)–(1.10).

5.3. *Remaining results in the rigid case*

Due to Proposition 5.4 one can identify each element $w \in RSG_n$ with the corresponding three-page tangle $\Gamma(w)$.

Definition 5.6. (the identity three-page tangle, knot-like three-tangles).

(a) *The identity* three-page tangle $\Gamma(1)$ consists of the parallel line segments joining marked points with same indices in every page P_i.
(b) A three-page tangle is called *knot-like*, if it contains a spatial graph added to the identity tangle $\Gamma(1)$. So, a knot-like tangle is encoded by a balanced word.

Theorem 1.6(c) follows from Lemma 5.7 stating that all balanced words encode all central elements of RSG_n. The algorithm to decide whether an element of RSG_n is balanced (or, equivalently, central) was discussed in Sec. 3.4.

Lemma 5.7. *An element* $w \in RSG_n$ *encodes a knot-like three-page tangle* $\Gamma(w)$ *if and only if the element* w *is central in the semigroup* RSG_n.

Proof. The part *only if* is geometrically evident: a spatial graph can be moved by a rigid isotopy to any place in a given tangle. Therefore, a balanced element commutes with any other element by Proposition 5.4.

For the part *if*, let w be a central element in RSG_n. Suppose that the associated three-page tangle $\Gamma(w)$ has an arc receding, in the page P_0 (say), to the left boundary of P_0. The same arc exists in the tangle $\Gamma(wa_1)$, but not in $\Gamma(a_1 w)$, because the pattern encoded by a_1 in Fig. 2 turns this arc from left to right.

Hence, the elements wa_1 and $a_1 w$ can not be equal in RSG_n, that is a contradiction. Then the word w is 0-balanced, similarly w is 1-balanced and 2-balanced. So, the word w is balanced and the three-page tangle $\Gamma(w)$ is knot-like. □

Proof of Corollary 1.9(a). If a spatial graph G is encoded by a word w_G, then its mirror image $\bar{G}$ is encoded by $\rho_n(w_G)$. So, Corollary 1.9a follows from Theorem 1.6(b). □

Proof of Corollary 1.10(a). Any element of the Dynnikov group $DG \subset DS = RSG_2 \subset RSG_n$ is invertible. Conversely, if an element $w \in RSG_n$ is invertible, then by Proposition 5.4 the corresponding three-page tangle $\Gamma(w)$ can not contain vertices of degree ≥ 3. Indeed, relations (1.1)–(1.10) preserve the number of the m-vertices of $\Gamma(w)$ or, equivalently, the number of the letters $x_{m,0}, x_{m,1}, x_{m,2}$ in w.

Hence, only the letters a_i, b_i, c_i, d_i may occur in w, i.e. $w \in RSG_2 = DS$. But the group of the invertible elements of DS is the group DG [5], i.e. $w \in DG$. □

5.4. *Non-rigid spatial graphs and spatial J-graphs*

Here the method of three-page embeddings will be extended to non-rigid spatial graphs and spatial J-graphs.

Proof of Theorem 1.7. The proof is similar to that of Theorem 1.6. Replace (1.9)–(1.10) by

$$x_{m,i}b_i(d_i^2 d_{i+1}^2 d_{i-1}^2) = x_{m,i}b_i, \quad \text{where } 3 \leq m \leq n,\ i \in \mathbb{Z}_3. \tag{1.9'}$$

Let us point out key moments. The non-rigid isotopy classes of graph tangles in $\{0 \leq z \leq 1\}$ form the semigroup NGT_n. As in Lemma 4.4 the semigroup NGT_n is generated by the letters of the alphabet $\mathbb{T}_n$, defining relations (4.1)–(4.12) and

$$\lambda_{m,k}\sigma_k = \lambda_{m,k}, \quad \text{where } 3 \leq m \leq n,\ k \in \mathbb{N}. \tag{4.13'}$$

New relations (4.13′) correspond to Reidemeister move $R5'$, which switches two arcs at an m-vertex in Fig. 1. The non-rigid isotopy classes of almost balanced tangles in $\{0 \leq z \leq 1\}$ form the semigroup NBT_n isomorphic to NGT_n. The isomorphism $\varphi : NGT_n \to NBT_n$ is defined also by formulae (5.1).

The non-rigid isotopy classes of three-page tangles in $\{0 \leq z \leq 1\}$ form the semigroup isomorphic to NSG_n, see Proposition 5.4 in Sec. 5.2. Actually, relations $\varphi(4.1)$–$\varphi(4.12)$ of NBT_n follow from relations (1.1)–(1.8) of NSG_n, see the proof of Lemma 5.5 in Sec. 6.3. New relations $\varphi(4.13')$ reduce to (1.9′) (the case of a $2q$-vertex is completely similar to the case of a $(2p-1)$-vertex):

$$\begin{aligned}
&\varphi(\lambda_{2p-1,k}\sigma_k^{-1}) \overset{(5.1)}{=} (d_2^{k-1}x_{2p-1,2}b_2^k)(d_2^k b_1 b_2 d_1 b_2^{k-1}) \overset{(1.2),(1.1)}{\sim} \\
&\overset{(1.2),(1.1)}{\sim} d_2^{k-1}x_{2p-1,2}(d_2 d_0)(d_0 d_1)d_1 b_2^{k-1} \overset{(1.2)}{\sim} d_2^{k-1}(x_{2p-1,2}b_2)(d_2^2 d_0^2 d_1^2)b_2^{k-1} \\
&\overset{(1.9')}{\sim} d_2^{k-1}(x_{2p-1,2}b_2)b_2^{k-1} \overset{(5.1)}{=} \varphi(\lambda_{2p-1,k}).
\end{aligned}$$

□

Corollaries 1.9(b) and 1.10(b) are verified absolutely analogously to Corollaries 1.9(a) and 1.10(a), respectively. The proof of Corollary 1.11 is contained in the proofs of the results for n-graphs: the condition $3 \leq m \leq n$ should be replaced by $m \in J$.

Proof of Corollary 1.12. Take a three-page embedding $G \subset \mathbb{Y}$. Let k_i be the number of the arcs of $G \cap P_i$. Split the page P_i into k_i pages. Let us move the arcs of $G \cap P_i$ to these new pages in such a way that each page contains exactly one arc.

Consider a plane $\mathbb{R}^2$ orthogonal to the axis α. Slightly deform the above arcs of G in $\mathbb{R}^3$ in such a way that their images under the projection $\mathbb{R}^3 \to \mathbb{R}^2$ along α become nonintersecting loops. This is a desired projection. □

6. Proof of Lemma 5.5

In Lemma 6.1 new word equivalences are deduced from relations (1.1)–(1.8) of the semigroup RSG_n. Lemmas 6.1, 6.2, 6.5, 6.6 will imply Lemma 6.7 on a decomposition of an i-balanced word. Relations $\varphi(4.1)$–$\varphi(4.13)$ reduce to relations (1.1)–(1.10) by Lemmas 6.7 and 6.8. Section 6.3 finishes the proof of Lemma 5.5 by exploiting Lemmas 6.8–6.10. All relations in this section will be verified formally, but they have a clear interpretation as well as relations (1.1)–(1.10) in Figs. 4–10.

6.1. *New word equivalences in the semigroup RSG_n*

Let $n \geq 2$ be fixed. The commutativity $uv \sim vu$ is denoted briefly by $u \leftrightarrow v$.

Lemma 6.1. *Relations* (1.1)–(1.8) *imply the following ones* (*where* $i \in \mathbb{Z}_3$ *and*

$$w_i \in \mathbb{B}_{n,i} = \{a_i,\ b_i,\ c_i,\ d_i,\ x_{m,i},\ b_{i-1}b_id_{i-1},\ b_{i-1}d_id_{i-1} \mid 3 \leq m \leq n\}):$$

$$b_i \sim d_{i+1}d_{i-1} \quad \text{or} \quad b_0 \sim d_1d_2,\ b_1 \sim d_2d_0,\ b_2 \sim d_0d_1; \tag{6.1}$$

$$d_i \sim b_{i-1}b_{i+1} \quad \text{or} \quad d_0 \sim b_2b_1,\ d_1 \sim b_0b_2,\ d_2 \sim b_1b_0; \tag{6.2}$$

$$d_{i+1}b_{i-1} \sim b_{i-1}d_{i+1}t_i,\ b_{i+1}d_{i-1} \sim t_id_{i-1}b_{i+1}, \quad \text{where } t_i = b_{i+1}d_{i-1}d_{i+1}b_{i-1}; \tag{6.3}$$

$$a_i \sim a_{i-1}b_{i+1}, \quad c_i \sim d_{i+1}c_{i-1}; \tag{6.4}$$

$$a_ib_i \sim a_{i-1}d_{i-1}, \quad d_ic_i \sim b_{i-1}c_{i-1}; \tag{6.5}$$

$$b_i \sim a_ib_ic_i, \quad d_i \sim a_id_ic_i; \tag{6.6}$$

$$b_i^{p-1}x_{2p-1,i}d_i^{p-1} \sim x_{2p-1,i+1}b_{i+1}; \tag{6.7}$$

$$d_i^{p-1}x_{2p-1,i+1}b_{i+1}b_i^p \sim x_{2p-1,i}b_i; \tag{6.8}$$

$$x_{2q,i+1} \sim d_{i-1}b_i^{q-2}x_{2q,i}d_i^{q-2}b_{i-1}; \tag{6.9}$$

$$b_i^{q-1}x_{2q,i}d_i^{q-1} \sim d_{i+1}x_{2q,i+1}b_{i+1}; \tag{6.10}$$

$$d_ic_i \leftrightarrow w_{i+1}; \tag{6.11}$$

$$b_ic_i \leftrightarrow w_{i-1}; \tag{6.12}$$

$$a_ib_i \leftrightarrow w_{i+1}; \tag{6.13}$$

$$a_id_i \leftrightarrow w_{i-1}; \tag{6.14}$$

$$t_i, t'_i \leftrightarrow w_i, \quad \textit{where } t_i = b_{i+1}d_{i-1}d_{i+1}b_{i-1},\ t'_i = d_{i-1}b_{i+1}b_{i-1}d_{i+1}; \tag{6.15}$$

$$x_{2p-1,i}b_i \leftrightarrow w_{i+1}; \tag{6.16}$$

$$d_i x_{2q,i} b_i \leftrightarrow w_{i+1}; \tag{6.17}$$

$$b_i^{p-1} x_{2p-1,i} d_i^{p-1} \leftrightarrow w_{i-1}; \tag{6.18}$$

$$d_{i-1}^{p-1} x_{2p-1,i} b_i b_{i-1}^{p} \leftrightarrow w_i; \tag{6.19}$$

$$b_i^{q-1} x_{2q,i} d_i^{q-1} \leftrightarrow w_{i-1}; \tag{6.20}$$

$$d_{i+1}b_{i-1}w_i d_{i-1}b_{i+1} \sim b_{i-1}d_{i+1}w_i b_{i+1}d_{i-1}; \tag{6.21}$$

$$b_{i-1}^2 a_i d_{i-1}^2 \sim (b_{i-1}a_i d_{i-1})d_i^2(b_{i-1}b_i d_{i-1})b_i; \tag{6.22}$$

$$b_{i-1}^2 b_i d_{i-1}^2 \sim (b_{i-1}b_i d_{i-1})d_i^2(b_{i-1}b_i d_{i-1})b_i; \tag{6.23}$$

$$b_{i-1}^2 c_i d_{i-1}^2 \sim d_i(b_{i-1}d_i d_{i-1})b_i^2(b_{i-1}c_i d_{i-1}); \tag{6.24}$$

$$b_{i-1}^2 d_i d_{i-1}^2 \sim d_i(b_{i-1}d_i d_{i-1})b_i^2(b_{i-1}d_i d_{i-1}); \tag{6.25}$$

$$b_{i-1}^2 x_{2p-1,i} d_{i-1}^2 \sim (b_{i-1}d_i d_{i-1})d_i x_{2p-1,i} b_i^2 (b_{i-1}b_i d_{i-1})(b_{i-1}^2 d_i d_{i-1}^2); \tag{6.26}$$

$$b_{i-1}^2 x_{2q,i} d_{i-1}^2 \sim (b_{i-1}^2 b_i d_{i-1}^2)(b_{i-1}d_i d_{i-1})d_i^2 x_{2q,i} b_i^2 (b_{i-1}b_i d_{i-1})(b_{i-1}^2 d_i d_{i-1}^2). \tag{6.27}$$

Proof. (6.1)–(6.3) follow from (1.1) and (1.2). Due to (1.2) we have $d_i \sim b_i^{-1}$, $b_{i-1}b_i d_{i-1} \sim (b_{i-1}d_i d_{i-1})^{-1}$ and $t'_i \sim t_i^{-1}$. Then (6.13) and (6.15)–(6.17) follow from (1.8). The other equivalences will be verified step by step exploiting the already checked ones. Since $i \in \mathbb{Z}_3 = \{0,1,2\}$, we have $(i+1)+1 = i-1$ and $(i-1)-1 = i+1$.

$$a_{i-1}b_{i+1} \overset{(1.3)}{\sim} (a_i d_{i+1})b_{i+1} \overset{(1.2)}{\sim} a_i, \quad d_{i+1}c_{i-1} \overset{(1.3)}{\sim} d_{i+1}(b_{i+1}c_i) \overset{(1.2)}{\sim} c_i; \tag{6.4}$$

$$a_i b_i \overset{(6.4)}{\sim} (a_{i-1}b_{i+1})b_i \overset{(6.2)}{\sim} a_{i-1}d_{i-1}, \quad d_i c_i \overset{(6.2)}{\sim} (b_{i-1}b_{i+1})c_i \overset{(1.3)}{\sim} b_{i-1}c_{i-1}; \tag{6.5}$$

$$\left.\begin{aligned} a_i b_i c_i &\overset{(6.5)}{\sim} a_i(d_{i+1}c_{i+1}) \overset{(1.3)}{\sim} a_{i-1}c_{i+1} \overset{(1.3)}{\sim} b_i, \\ a_i d_i c_i &\overset{(6.5)}{\sim} a_i(b_{i-1}c_{i-1}) \overset{(6.4)}{\sim} a_{i+1}c_{i-1} \overset{(1.3)}{\sim} d_i; \end{aligned}\right\} \tag{6.6}$$

$$\left.\begin{aligned} &b_i^{p-1}x_{2p-1,i}d_i^{p-1} \overset{(1.4)}{\sim} b_i^{p-1}(d_i^{p-1}x_{2p-1,i+1}d_{i-1}b_i^{p-2})d_i^{p-1} \overset{(1.2)}{\sim} \\ &\overset{(1.2)}{\sim} x_{2p-1,i+1}(d_{i-1}d_i) \overset{(6.1)}{\sim} x_{2p-1,i+1}b_{i+1}; \end{aligned}\right\} \tag{6.7}$$

$$d_i^{p-1}(x_{2p-1,i+1}b_{i+1})b_i^p \overset{(6.7)}{\sim} d_i^{p-1}(b_i^{p-1}x_{2p-1,i}d_i^{p-1})b_i^p \overset{(1.2)}{\sim} x_{2p-1,i}b_i; \tag{6.8}$$

$$\left.\begin{aligned} &d_{i-1}b_i^{q-2}x_{2q,i}d_i^{q-2}b_{i-1} \overset{(1.4)}{\sim} \\ &\overset{(1.4)}{\sim} d_{i-1}b_i^{q-2}(d_i^{q-2}b_{i-1}x_{2q,i+1}d_{i-1}b_i^{q-2})d_i^{q-2}b_{i-1} \overset{(1.2)}{\sim} x_{2q,i+1}; \end{aligned}\right\} \tag{6.9}$$

$$\left.\begin{aligned} &b_i^{q-1}x_{2q,i}d_i^{q-1} \overset{(1.4)}{\sim} b_i^{q-1}(d_i^{q-2}b_{i-1}x_{2q,i+1}d_{i-1}b_i^{q-2})d_i^{q-1} \overset{(1.2)}{\sim} \\ &(b_i b_{i-1})x_{2q,i+1}(d_{i-1}d_i) \overset{(6.1)}{\sim} (b_i b_{i-1})x_{2q,i+1}b_{i+1} \overset{(6.2)}{\sim} d_{i+1}x_{2q,i+1}b_{i+1}. \end{aligned}\right\} \tag{6.10}$$

Below in the proof of (6.11) we commute firstly b_{i+1} with d_ic_i and after that we use this equivalence to commute a_{i+1} with d_ic_i.

$$\left.\begin{array}{c} b_{i+1}(d_ic_i) \overset{(6.6)}{\sim} (a_{i+1}b_{i+1}c_{i+1})(d_ic_i) \overset{(1.7)}{\sim} a_{i+1}b_{i+1}(d_ic_i)c_{i+1} \overset{(6.2)}{\sim} \\ \overset{(6.2)}{\sim} a_{i+1}b_{i+1}(b_{i-1}b_{i+1})c_ic_{i+1} \overset{(1.3)}{\sim} (a_{i+1}b_{i+1})b_{i-1}c_{i-1}c_{i+1} \overset{(1.8)}{\sim} \\ \overset{(1.8)}{\sim} b_{i-1}c_{i-1}(a_{i+1}b_{i+1})c_{i+1} \overset{(6.6)}{\sim} b_{i-1}c_{i-1}b_{i+1} \overset{(1.3)}{\sim} \\ \overset{(1.3)}{\sim} b_{i-1}(b_{i+1}c_i)b_{i+1} \overset{(6.2)}{\sim} (d_ic_i)b_{i+1}; \end{array}\right\} \quad (6.11b)$$

$$\left.\begin{array}{c} a_{i+1}(d_ic_i) \overset{(1.3)}{\sim} (a_{i-1}d_i)(d_ic_i) \overset{(6.2)}{\sim} a_{i-1}(b_{i-1}b_{i+1})(d_ic_i) \overset{(6.11b)}{\sim} \\ \overset{(6.11b)}{\sim} a_{i-1}b_{i-1}(d_ic_i)b_{i+1} \overset{(6.13)}{\sim} (d_ic_i)(a_{i-1}b_{i-1})b_{i+1} \overset{(6.2)}{\sim} \\ \overset{(6.2)}{\sim} (d_ic_i)(a_{i-1}d_i) \overset{(1.3)}{\sim} (d_ic_i)a_{i+1}. \end{array}\right\} \quad (6.11a)$$

The remaining equivalences in (6.11) follow from (6.11a), (6.11b) and (1.7). Equivalences (6.12) are easily proved by (6.5) and (6.11), as well as (6.14) by (6.5) and (6.13), as well as (6.18) by (6.7) and (6.16), as well as (6.19) by (6.8) and (6.16), as well as (6.20) by (6.10) and (6.17). The last calculations are straightforward:

$$\left.\begin{array}{c} d_{i+1}b_{i-1}w_id_{i-1}b_{i+1} \overset{(6.3)}{\sim} (b_{i-1}d_{i+1}t_i)w_id_{i-1}b_{i+1} \overset{(6.15)}{\sim} \\ \overset{(6.15)}{\sim} b_{i-1}d_{i+1}(w_it_i)d_{i-1}b_{i+1} \overset{(6.3)}{\sim} b_{i-1}d_{i+1}w_ib_{i+1}d_{i-1}; \end{array}\right\} \quad (6.21)$$

$$\left.\begin{array}{c} b_{i-1}^2a_id_{i-1}^2 \overset{(1.2)}{\sim} b_{i-1}^2a_i(d_ib_i)d_{i-1}^2 \overset{(6.14)}{\sim} b_{i-1}(a_id_i)(b_{i-1}b_i)d_{i-1}^2 \overset{(6.2)}{\sim} \\ \overset{(6.2)}{\sim} b_{i-1}a_id_i(b_{i-1}b_i)d_{i-1}(b_{i+1}b_i) \overset{(6.15)}{\sim} b_{i-1}a_ib_{i+1}(d_ib_{i-1}b_id_{i-1})b_i \overset{(6.1)}{\sim} \\ \overset{(6.1)}{\sim} (b_{i-1}a_id_{i-1})d_i^2(b_{i-1}b_id_{i-1})b_i; \end{array}\right\} \quad (6.22)$$

$$\left.\begin{array}{c} b_{i-1}^2b_id_{i-1}^2 \overset{(1.2)}{\sim} b_{i-1}(b_id_i)b_{i-1}b_id_{i-1}^2 \overset{(6.2)}{\sim} b_{i-1}b_i(d_ib_{i-1}b_id_{i-1})(b_{i+1}b_i) \\ \overset{(6.15)}{\sim} b_{i-1}b_ib_{i+1}(d_ib_{i-1}b_id_{i-1})b_i \overset{(6.1)}{\sim} (b_{i-1}b_id_{i-1})d_i^2(b_{i-1}b_id_{i-1})b_i; \end{array}\right\} \quad (6.23)$$

$$\left.\begin{array}{c} b_{i-1}^2c_id_{i-1}^2 \overset{(1.2)}{\sim} b_{i-1}^2(d_ib_i)c_id_{i-1}^2 \overset{(6.12)}{\sim} b_{i-1}^2d_id_{i-1}(b_ic_i)d_{i-1} \overset{(6.1)}{\sim} \\ \overset{(6.1)}{\sim} (d_id_{i+1})(b_{i-1}d_id_{i-1}b_i)c_id_{i-1} \overset{(6.15)}{\sim} d_i(b_{i-1}d_id_{i-1}b_i)d_{i+1}c_id_{i-1} \overset{(6.2)}{\sim} \\ \overset{(6.2)}{\sim} d_i(b_{i-1}d_id_{i-1})b_i^2(b_{i-1}c_id_{i-1}); \end{array}\right\} \quad (6.24)$$

$$\left.\begin{array}{c} b_{i-1}^2d_id_{i-1}^2 \overset{(1.2)}{\sim} b_{i-1}^2d_id_{i-1}(b_id_i)d_{i-1} \overset{(6.1)}{\sim} (d_id_{i+1})b_{i-1}d_id_{i-1}b_id_id_{i-1} \\ \overset{(6.15)}{\sim} d_i(b_{i-1}d_id_{i-1}b_i)d_{i+1}d_id_{i-1} \overset{(6.2)}{\sim} d_i(b_{i-1}d_id_{i-1})b_i^2(b_{i-1}d_id_{i-1}); \end{array}\right\} \quad (6.25)$$

$$\left.\begin{array}{c} b_{i-1}^2 x_{2p-1,i} d_{i-1}^2 \overset{(1.2)}{\sim} b_{i-1}^2 x_{2p-1,i}(b_i b_{i+1}^2 d_{i+1}^2 d_i) d_{i-1}^2 \\ \overset{(6.16)}{\sim} b_{i-1}^2 b_{i+1}^2 (x_{2p-1,i} b_i) d_{i+1}^2 d_i d_{i-1}^2 \overset{(6.2)}{\sim} \\ \overset{(6.2)}{\sim} b_{i-1}(d_i d_{i+1}) b_{i+1}^2 x_{2p-1,i} b_i (b_i b_{i-1})^2 d_i d_{i-1}^2 \overset{(1.2)}{\sim} \\ \overset{(1.2)}{\sim} b_{i-1} d_i b_{i+1} x_{2p-1,i} b_i^2 (b_{i-1} b_i) b_{i-1} d_i d_{i-1}^2 \overset{(6.1),(1,2)}{\sim} \\ (b_{i-1} d_i d_{i-1}) d_i x_{2p-1,i} b_i^2 (b_{i-1} b_i d_{i-1})(b_{i-1}^2 d_i d_{i-1}^2); \end{array}\right\} \quad (6.26)$$

$$\left.\begin{array}{c} b_{i-1}^2 x_{2q,i} d_{i-1}^2 \overset{(1.2)}{\sim} b_{i-1}^2 (b_i d_i) x_{2q,i} (b_i b_{i+1}^2 d_{i+1}^2 d_i) d_{i-1}^2 \overset{(6.17)}{\sim} \\ b_{i-1}^2 b_i b_{i+1}^2 (d_i x_{2q,i} b_i) d_{i+1}^2 d_i d_{i-1}^2 \overset{(6.1)}{\sim} b_{i-1}^2 b_i (d_{i-1} d_i)^2 d_i x_{2q,i} b_i d_{i+1}^2 d_i d_{i-1}^2 \\ \overset{(6.2)}{\sim} (b_{i-1}^2 b_i d_{i-1}) d_i d_{i-1} d_i^2 x_{2q,i} b_i (b_i b_{i-1})^2 d_i d_{i-1}^2 \overset{(1.2)}{\sim} \\ (b_{i-1}^2 b_i d_{i-1}^2)(b_{i-1} d_i d_{i-1}) d_i^2 x_{2q,i} b_i^2 (b_{i-1} b_i d_{i-1})(b_{i-1}^2 d_i d_{i-1}^2). \end{array}\right\} \quad (6.27)$$

□

6.2. *Decomposition of i-balanced words*

The goal here is to prove Lemma 6.7, which allows us to decompose i-balanced words into elementary ones. Due to this decomposition infinitely many relations $\varphi(4.1)$–$\varphi(4.13)$ will reduce to finitely many relations (1.1)–(1.10).

Lemma 6.2. *For each $i \in \mathbb{Z}_3$, any i-balanced word is equivalent by* (1.1)–(1.8), (6.1)–(6.27) *to an i-balanced word containing only a_i, b_i, c_i, d_i, $x_{m,i}$, b_{i-1}, d_{i-1}.*

Proof. The rest letters can be eliminated by using the substitutions:

$$x_{2p-1,i+1} \overset{(1.4)}{\sim} d_{i+1}^{p-1} x_{2p-1,i-1} d_i b_{i+1}^{p-2}, \quad x_{2p-1,i-1} \overset{(1.4)}{\sim} d_{i-1}^{p-1} x_{2p-1,i} d_i b_{i-1}^{p-2},$$
$$x_{2q,i-1} \overset{(1.4)}{\sim} d_{i-1}^{q-2} b_{i+1} x_{2q,i} d_{i+1} b_i^{q-2}, \quad x_{2q,i+1} \overset{(6.9)}{\sim} d_{i-1} b_i^{q-2} x_{2q,i} d_i^{q-2} b_{i-1},$$
$$a_{i+1} \overset{(1.3)}{\sim} a_{i-1} d_i, \quad c_{i+1} \overset{(1.3)}{\sim} b_i c_{i-1}, \quad a_{i-1} \overset{(1.3)}{\sim} a_i d_{i+1}, \quad c_{i-1} \overset{(1.3)}{\sim} b_{i+1} c_i,$$
$$b_{i+1} \overset{(6.1)}{\sim} d_{i-1} d_i, \qquad d_{i+1} \overset{(6.2)}{\sim} b_i b_{i-1}.$$

□

In what follows, fix an index $i \in \mathbb{Z}_3$.

Definition 6.3 (the encoding $\mu(w)$ and depth $d(w)$ of a word).

(a) Let w be an i-balanced word in the letters a_i, b_i, c_i, d_i, $x_{m,i}$, b_{i-1}, d_{i-1}. Consider the following substitution

$$\mu : a_i, b_i, c_i, d_i, x_{m,i} \to \bullet; \quad b_{i-1} \to (; \quad d_{i-1} \to).$$

Denote by $\mu(w)$ the resulting *encoding* consisting of brackets and bullets.

(b) Since w is i-balanced, the encoding $\mu(w)$ without bullets is a balanced bracket expression, see Definition 3.3. For each place k, denote by $dif(k)$ the difference between the number of the left and right brackets in the subword of $\mu(w)$ ending at this place. The maximum of $dif(k)$ over all k is called *the depth* of w, $d(w)$.

For example, the word $w = b_{i-1}^2 a_i d_{i-1}^2$ gives $\mu(w) = ((\bullet))$ and $d(w) = 2$.

Definition 6.4 (stars and star decomposable words).

(a) *A star of depth* k is an encoding of the type $({}^k\bullet)^k$ that has k pairs of brackets. The bullet $\bullet$ is the star of depth 0.
(b) If the encoding $\mu(w)$ decomposes into several stars, then w is called *star decomposable*. In this case, the depth $d(w)$ is the maximal depth over the depths of all stars participating in the decomposition.

Lemma 6.5. *Every* i*-balanced word* w *is equivalent to a star decomposable word* w' *of same depth* $d(w') = d(w)$.

Proof. Consider the beginning of the encoding $\mu(w)$. After several initial left brackets the encoding $\mu(w)$ contains either a right bracket or a bullet.

In the first case, delete a pair of brackets () by relation $b_{i-1}d_{i-1} \overset{(1.2)}{\sim} \emptyset$. Hence, we may assume that the next symbol after k left brackets is a bullet. Since $\mu(w)$ is balanced, after this bullet there can be a sequence of j, $0 \le j \le k$, right brackets. If $j < k$, then insert the subword $d_{i-1}^{k-j}b_{i-1}^{k-j} \overset{(1.2)}{\sim} \emptyset$ in w after the last right bracket. This operation does not change the depth $d(w)$. Therefore, in the resulting word w_1, the encoding $\mu(w_1)$ contains a star of depth k at the beginning.

For instance, starting with word $w = b_{i-1}a_i^2d_{i-1}$, we get $w_1 = b_{i-1}a_id_{i-1}b_{i-1}a_id_{i-1}$ and $\mu(w_1) = (\bullet)(\bullet)$. Continuing this process, after a finite number of steps, we get a star decomposable word w_N with $d(w_N) = d(w)$. $\square$

For a letter s, denote by s' the word $b_{i-1}sd_{i-1}$, for example, $a_i' = b_{i-1}a_id_{i-1}$.

Lemma 6.6. *For each* $i \in \mathbb{Z}_3$, *every star decomposable word* w *is equivalent to a word decomposed into the following* i*-balanced subwords*

$$a_i,\ b_i,\ c_i,\ d_i,\ x_{m,i},\ a_i',\ b_i',\ c_i',\ d_i',\ x_{m,i}',\ \textit{where}\ \ 3 \le m \le n.$$

Proof. Induction on $d(w)$. The case $d(w) = 1$ is trivial. Let the encoding $\mu(w)$ contain a star of depth ≥ 2. Apply one of the following moves to every such star.

$$u = b_{i-1}^2a_id_{i-1}^2 \overset{(6.22)}{\sim} a_i'd_i^2b_i'b_i = v, \quad \text{i.e. } \mu(u) = ((\bullet)) \to \mu(v) = (\bullet)\bullet\bullet(\bullet)\bullet;$$

$$u = b_{i-1}^2b_id_{i-1}^2 \overset{(6.23)}{\sim} b_i'd_i^2b_i'b_i = v, \quad \text{i.e. } \mu(u) = ((\bullet)) \to \mu(v) = (\bullet)\bullet\bullet(\bullet)\bullet;$$

$$u = b_{i-1}^2c_id_{i-1}^2 \overset{(6.24)}{\sim} d_id_i'b_i^2c_i' = v, \quad \text{i.e. } \mu(u) = ((\bullet)) \to \mu(v) = \bullet(\bullet)\bullet\bullet(\bullet);$$

$$u = b_{i-1}^2d_id_{i-1}^2 \overset{(6.25)}{\sim} d_id_i'b_i^2d_i' = v, \quad \text{i.e. } \mu(u) = ((\bullet)) \to \mu(v) = \bullet(\bullet)\bullet\bullet(\bullet);$$

$$u = b_{i-1}^2x_{2p-1,i}d_{i-1}^2 \overset{(6.26)}{\sim} (b_{i-1}d_id_{i-1})d_ix_{2p-1,i}b_i^2(b_{i-1}b_id_{i-1})(b_{i-1}^2d_id_{i-1}^2) \overset{(6.25)}{\sim}$$
$$b_i'd_ix_{2p-1,i}b_i^2b_i'(d_id_i'b_i^2d_i') = v, \text{ i.e. } \mu(u) = ((\bullet)) \to \mu(v) = (\bullet)\bullet\bullet\bullet\bullet(\bullet)\bullet(\bullet)\bullet\bullet(\bullet);$$

$$u = b_{i-1}^2 x_{2q,i} d_{i-1}^2 \overset{(6.27)}{\sim} (b_{i-1}^2 b_i d_{i-1}^2)(b_{i-1} d_i d_{i-1}) d_i^2 x_{2q,i} b_i^2 (b_{i-1} b_i d_{i-1})(b_{i-1}^2 d_i d_{i-1}^2)$$
$$\overset{(6.23),(6.25)}{\sim} (b_i' d_i^2 b_i' b_i) d_i' d_i^2 x_{2q,i} b_i^2 b_i' (d_i d_i' b_i^2 d_i') = v, \text{ i.e.}$$

$$\mu(u) = ((\bullet)) \to \mu(v) = (\bullet) \bullet \bullet (\bullet) \bullet (\bullet) \bullet \bullet \bullet \bullet \bullet (\bullet) \bullet (\bullet) \bullet \bullet (\bullet).$$

We get a word $w_1 \sim w$ of depth $d(w_1) = d(w) - 1$. By Lemma 6.5 the word w_1 is equivalent to a star decomposable word w_2 of depth $d(w_2) = d(w_1) = d(w) - 1$. The induction step is complete. □

Recall that $W_{n,i}$ is the set of all i-balanced words in the alphabet $\mathbb{A}_n$.

Lemma 6.7. *For each $i \in \mathbb{Z}_3$, every i-balanced word from $W_{n,i}$ is equivalent to a word decomposed into i-balanced words belonging to the set*

$$\mathbb{B}_{n,i} = \{a_i,\ b_i,\ c_i,\ d_i,\ b_{i-1} b_i d_{i-1},\ b_{i-1} d_i d_{i-1},\ x_{m,i} \mid 3 \le m \le n\}.$$

Proof. By Lemmas 6.5 and 6.6 it remains to eliminate only the following words:

$$a_i' = b_{i-1} a_i d_{i-1} \overset{(6.1),(1.2)}{\sim} (d_i d_{i+1}) a_i d_{i-1} \overset{(1.2)}{\sim} (d_i d_{i+1}) a_i (b_i d_i) d_{i-1} \overset{(6.13)}{\sim}$$
$$\overset{(6.13)}{\sim} d_i (a_i b_i) d_{i+1} d_i d_{i-1} \overset{(6.2)}{\sim} d_i a_i b_i^2 (b_{i-1} d_i d_{i-1});$$

$$c_i' = b_{i-1} c_i d_{i-1} \overset{(6.2),(1.2)}{\sim} b_{i-1} c_i (b_{i+1} b_i) \overset{(1.2)}{\sim} b_{i-1} (b_i d_i c_i)(b_{i+1} b_i) \overset{(6.11)}{\sim}$$
$$\overset{(6.11)}{\sim} b_{i-1} b_i b_{i+1} (d_i c_i) b_i \overset{(6.1)}{\sim} (b_{i-1} b_i d_{i-1}) d_i^2 c_i b_i;$$

$$x_{2p-1,i}' = b_{i-1} x_{2p-1,i} d_{i-1} \overset{(1.2)}{\sim} b_{i-1} x_{2p-1,i} (b_i d_i) d_{i-1} \overset{(6.1)}{\sim} (d_i d_{i+1})(x_{2p-1,i} b_i) d_i d_{i-1}$$
$$\overset{(6.16)}{\sim} d_i (x_{2p-1,i} b_i) d_{i+1} d_i d_{i-1} \overset{(6.2)}{\sim} d_i x_{2p-1,i} b_i^2 (b_{i-1} d_i d_{i-1});$$

$$x_{2q,i}' = b_{i-1} x_{2q,i} d_{i-1} \overset{(1.2)}{\sim} b_{i-1} (b_i d_i) x_{2q,i} (b_i d_i) d_{i-1}$$
$$\overset{(1.2)}{\sim} b_{i-1} b_i (b_{i+1} d_{i+1})(d_i x_{2q,i} b_i) d_i d_{i-1} \overset{(6.17)}{\sim}$$
$$b_{i-1} b_i b_{i+1} (d_i x_{2q,i} b_i) d_{i+1} d_i d_{i-1} \overset{(6.1),(6.2)}{\sim} (b_{i-1} b_i d_{i-1}) d_i^2 x_{2q,i} b_i^2 (b_{i-1} d_i d_{i-1}).$$

In equivalences (6.11)–(6.21), let us replace the condition $w_i \in \mathbb{B}_{n,i}$ by w_i. The resulting relations will be denoted by $(6.11')$–$(6.21')$. □

Lemma 6.8. *Relations* $(6.11')$–$(6.21')$ *hold for all i-balanced words w_i.*

Proof. By Lemma 6.7 each i-balanced word $w \in W_{n,i}$ can be decomposed into the i-balanced words belonging to $\mathbb{B}_{n,i}$. Since commutative equivalences (6.11)–(6.21) hold for words in $\mathbb{B}_{n,i}$ by Lemma 6.7, they also hold for all words $w_i \in W_{n,i}$. □

6.3. *Deduction of relations* $\varphi(4.1)$–$\varphi(4.13)$ *from* (1.1)–(1.10)

Relations $\varphi(4.1)$–$\varphi(4.13)$ between words in $\mathbb{A}_n$ were obtained from (4.1)–(4.13) under the isomorphism $\varphi : RGT_n \to RBT_n$, see Sec. 5.1.

For $l \geq 1$, denote by u_l a generator of the semigroup RGT_n, i.e.

$$u_l \in \{\xi_l, \eta_l, \sigma_l, \sigma_l^{-1}, \lambda_{m,l} \mid 3 \leq m \leq n\}.$$

Define *the shift maps* $\theta_k : RGT_n \to RGT_n$ and $\omega_k : RBT_n \to RBT_n$ by $\theta_k(u_l) = u_{k+l}$ and $\omega_k(w) = d_2^k w b_2^k$, $k \geq 1$. Then θ_k is a well-defined homomorphism of semigroups. Indeed, each relation in (4.1)–(4.13) for $k > 1$ is obtained from the corresponding relation for $k = 1$ by the shift map θ_{k-1}. For example, relation $\xi_k\xi_l = \xi_{l+2}\xi_k$ is obtained from $\xi_1\xi_{l-k+1} = \xi_{l-k+3}\xi_1$ under the shift map θ_{k-1}.

Due to relations (1.2) the shift map ω_k sends equivalent words to equivalent ones, i.e. ω_k is also a homomorphism. The following diagram is commutative.

$$\begin{array}{ccc} RGT_n & \xrightarrow{\theta_k} & RGT_n \\ \varphi\downarrow & & \downarrow\varphi \\ RBT_n & \xrightarrow{\omega_k} & RBT_n \end{array}$$

Lemma 6.9. *For each* $k \in \mathbb{N}$, *relations* $\varphi(4.1)$–$\varphi(4.13)$ *can be obtained from relations* $\varphi(4.1)$–$\varphi(4.13)$ *for* $k = 1$ *by using relation* (1.2) $b_2d_2 \sim 1 \sim d_2b_2$.

Proof. It follows from the commutativity of the diagram. For instance, we have

$$\varphi(\xi_k\xi_l) = \varphi \circ \theta_{k-1}(\xi_1\xi_{l-k+1}) = \omega_{k-1} \circ \varphi(\xi_1\xi_{l-k+1}) = d_2^{k-1}\varphi(\xi_1\xi_{l-k+1})b_2^{k-1} \overset{\varphi(4.1)}{\sim} d_2^{k-1}\varphi(\xi_{l-k+3}\xi_1)b_2^{k-1} = \omega_{k-1} \circ \varphi(\xi_{l-k+3}\xi_1) = \varphi \circ \theta_{k-1}(\xi_{l-k+3}\xi_1) = \varphi(\xi_{l+2}\xi_k).$$

□

Lemma 6.10. *Under the map* $\varphi : RGT_n \to RBT_n$ *relations* (1.1)–(1.10), (6.1)–(6.10), (6.11′)–(6.21′) *imply the following ones*:

$$\varphi(\Sigma_{1,l}) \sim b_1d_2^l d_1b_2^l, \qquad \textit{where } \Sigma_{k,l} = \sigma_k\sigma_{k+1}\cdots\sigma_{k+l-1}; \tag{6.28}$$

$$\varphi(\bar{\Sigma}_{k,l}) \sim d_2^{k-1}b_1^l d_2 d_1^l b_2^k, \quad \textit{where } \bar{\Sigma}_{k,l} = \sigma_{k+l-1}\cdots\sigma_{k+1}\sigma_k; \tag{6.29}$$

$$\varphi(\Sigma'_{1,l}) \sim D_{l+1,2}, \qquad \textit{where } D_{k,i} = d_i^k d_{i+1}^k d_{i-1}^k. \tag{6.30}$$

Proof. The following calculations are straightforward:

$$\left.\begin{aligned} \varphi(\Sigma_{1,l}) = \varphi(\sigma_1)\cdots\varphi(\sigma_l) &\overset{(5.1)}{\sim} (b_1d_2d_1b_2)(d_2b_1d_2d_1b_2^2)\cdots(d_2^{l-1}b_1d_2d_1b_2^l) \\ &\overset{(1.2)}{\sim} (b_1d_2d_1)(b_1d_2d_1)(b_1d_2d_1b_2^l) \overset{(1.2)}{\sim} b_1d_2^l d_1b_2^l; \end{aligned}\right\} \tag{6.28}$$

$$\left.\begin{aligned}\varphi(\bar\Sigma_{1,l}) &\overset{(5.1)}{\sim} (d_2^{l-1}b_1d_2d_1b_2^l)(d_2^{l-2}b_1d_2d_1b_2^{l-1})\cdots(b_1d_2d_1b_2)\\ &\overset{(1.2)}{\sim} d_2^{l-1}(b_1d_2d_1b_2^2)^{l-2}(b_1d_2d_1b_2)(b_2b_1)d_2d_1b_2\\ &\overset{(6.15')}{\sim} d_2^{l-1}(b_1d_2d_1b_2^2)^{l-2}(b_2b_1)(b_1d_2d_1b_2)d_2d_1b_2\\ &\overset{(1.2)}{\sim} d_2^{l-1}(b_1d_2d_1b_2^2)^{l-3}(b_1d_2d_1b_2)(b_2^2b_1^2)d_2d_1^2b_2\\ \overset{(6.15'),(1.2)}{\sim}\cdots &\overset{(6,15'),(1.2)}{\sim} d_2^{l-1}(b_2^{l-1}b_1^l)d_2d_1^lb_2 \overset{(1.2)}{\sim} b_1^ld_2d_1^lb_2,\\ \varphi(\bar\Sigma_{k,l}) = \varphi(\theta_{k-1}(\bar\Sigma_{1,l})) &= \omega_{k-1}(\varphi(\bar\Sigma_{1,l})) \sim d_2^{k-1}b_1^ld_2d_1^lb_2^k;\end{aligned}\right\}\quad(6.29)$$

$$\left.\begin{aligned}&\varphi(\Sigma'_{1,l}) = \varphi(\sigma_l^{-1})\cdots\varphi(\sigma_1^{-1}\cdots\sigma_l^{-1}) = \varphi(\bar\Sigma_{l,1}^{-1})\cdots\varphi(\bar\Sigma_{2,l-1}^{-1})\varphi(\bar\Sigma_{1,l}^{-1})\\ &\overset{(6.29)}{\sim} (d_2^lb_1b_2d_1b_2^{l-1})(d_2^{l-1}b_1b_2d_1b_2^{l-2})\cdots(d_2^2b_1^{l-1}b_2d_1^{l-1}b_2)(d_2b_1^lb_2d_1^l) \overset{(1.2)}{\sim}\\ &\overset{(1.2)}{\sim} d_2^lb_1(b_2b_1)^{l-1}b_2d_1^l \overset{(6.2)}{\sim} d_2^lb_1d_0^{l-1}b_2d_1^l \overset{(6.1)}{\sim} d_2^{l+1}d_0^{l+1}d_1^{l+1} = D_{l+1,2}.\end{aligned}\right\}\quad(6.30)$$

□

Proof of Lemma 5.5. Here relations $\varphi(4.1)$–$\varphi(4.13)$ are deduced from equivalences (1.1)–(1.10), (6.1)–(6.10), (6.11′)–(6.21′), (6.28)–(6.30). Denote by the star $\star$ the following images under the map $\varphi : RGT_n \to RBT_n$, see (5.1):

$$\begin{aligned}&\varphi(\xi_1) = d_2c_2, &&\varphi(\sigma_1) = b_1d_2d_1b_2, &&\varphi(\lambda_{2p-1,1}) = x_{2p-1,2}b_2,\\ &\varphi(\eta_1) = a_2b_2, &&\varphi(\sigma_1^{-1}) = d_2b_1b_2d_1, &&\varphi(\lambda_{2q,1}) = d_2x_{2q,2}b_2.\end{aligned}$$

The words $\varphi(u_l) = d_2^{l-1}\star b_2^{l-1}$ are 1-balanced (see Fig. 14), i.e. $d_2^l\star b_2^l \in W_{n,1}$ for each $l\in\mathbb{N}$. Then $\varphi(4.1)$–$\varphi(4.4)$ can be proved following the same scheme:

$$\varphi(\xi_1u_l) \overset{(1.2)}{\sim} d_2^2(b_2c_2)(d_2^{l-1}\star b_2^{l-1}) \overset{(6.12')}{\sim} d_2^2(d_2^{l-1}\star b_2^{l-1})(b_2c_2) \overset{(1.2)}{\sim} \varphi(u_{l+2}\xi_1);\quad(4.1)$$

$$\varphi(\eta_1u_l) \overset{(1.2)}{\sim} (a_2d_2)(d_2^{l-3}\star b_2^{l-3})b_2^2 \overset{(6.14')}{\sim} (d_2^{l-3}\star b_2^{l-3})(a_2d_2)b_2^2 \overset{(1.2)}{\sim} \varphi(u_{l-2}\eta_1);\quad(4.2)$$

$$\left.\begin{aligned}&\varphi(\sigma_1u_l) \overset{(1.2)}{\sim} (b_1d_2d_1)(d_2^{l-2}\star b_2^{l-1}) \overset{(6.1)}{\sim} d_2^2(b_2d_0d_2b_0)(d_2^{l-3}\star b_2^{l-3})b_2^2 \overset{(6.15')}{\sim}\\ &d_2^2(d_2^{l-3}\star b_2^{l-3})(b_2d_0d_2b_0)b_2^2 \overset{(6.2)}{\sim} (d_2^{l-1}\star b_2^{l-2})(b_2b_1)d_2d_1b_2 \overset{(5.1)}{=} \varphi(u_l\sigma_1);\end{aligned}\right\}\quad(4.3)$$

$$\left.\begin{aligned}&\varphi(\lambda_{2p-1,1}u_l) \overset{(1.2)}{\sim} d_2^{p-1}(b_2^{p-1}x_{2p-1,2}d_2^{p-1})(d_2^{l-p-1}\star b_2^{l-p-1})b_2^p \overset{(6.18')}{\sim}\\ &d_2^{p-1}(d_2^{l-p-2}\star b_2^{l-p-2})(b_2^{p-1}x_{2p-1,2}d_2^{p-1})b_2^p \overset{(1.2)}{\sim} \varphi(u_{l-1}\lambda_{2p-1,1});\\ &\varphi(\lambda_{2q,1}u_l) \overset{(1.2)}{\sim} d_2^q(b_2^{q-1}x_{2q,2}d_2^{q-1})(d_2^{l-q-1}\star b_2^{l-q-1})b_2^q \overset{(6.20')}{\sim}\\ &d_2^q(d_2^{l-q-1}\star b_2^{l-q-1})(b_2^{q-1}x_{2q,2}d_2^{q-1})b_2^q \overset{(1.2)}{\sim} \varphi(u_l\lambda_{2q,1}).\end{aligned}\right\}\quad(4.4)$$

The remaining calculations are straightforward:

$$\varphi(\eta_2\xi_1) \overset{(1.2)}{\sim} d_2(a_2b_2c_2) \overset{(6.6)}{\sim} d_2b_2 \overset{(1.2)}{\sim} 1 \overset{(1.2),(6.6)}{\sim} (a_2d_2c_2)b_2 \overset{(1.2)}{\sim} \varphi(\eta_1\xi_2);\quad(4.5)$$

$$\left.\begin{array}{c}\varphi(\eta_3\sigma_2\xi_1) \overset{(5.1)}{=} (d_2^2a_2b_2^3)(d_2b_1d_2d_1b_2^2)(d_2c_2) \overset{(1.2)}{\sim}\\ \overset{(1.2)}{\sim} d_2^2a_2b_2(b_2b_1)d_2d_1(b_2c_2) \overset{(6.2)}{\sim} d_2^2(a_2b_2)d_0d_2d_1(b_2c_2) \overset{(6.13)}{\sim}\\ \overset{(6.13)}{\sim} d_2^2d_0(a_2b_2)d_2d_1(b_2c_2) \overset{(6.12)}{\sim} d_2(d_2d_0)(a_2b_2)d_2(b_2c_2)d_1 \overset{(1.2)}{\sim}\\ \overset{(1.2)}{\sim} d_2(d_2d_0)(a_2b_2c_2)d_1 \overset{(6.1)}{\sim} d_2b_1(a_2b_2c_2)d_1 \overset{(6.6)}{\sim} d_2b_1b_2d_1 \overset{(5.1)}{=} \varphi(\sigma_1^{-1}),\\ \varphi(\eta_1\sigma_2\xi_3) \overset{(5.1)}{=} (a_2b_2)(d_2b_1d_2d_1b_2^2)(d_2^3c_2b_2^2) \overset{(1.2)}{\sim} a_2b_1d_2(d_1d_2)c_2b_2^2\\ \overset{(6.1)}{\sim} a_2b_1d_2(b_0c_2)b_2^2 \overset{(1.3)}{\sim} (a_0d_1)b_1d_2c_1b_2^2 \overset{(1.2)}{\sim} a_0d_2(b_1d_1)c_1b_2^2 \overset{(6.11)}{\sim}\\ a_0d_2b_1b_2(d_1c_1)b_2 \overset{(6.15)}{\sim} (d_2b_1b_2d_1)(a_0c_1)b_2 \overset{(1.3),(1.2)}{\sim} d_2b_1b_2d_1 \overset{(5.1)}{=} \varphi(\sigma_1^{-1});\end{array}\right\} \quad (4.6)$$

$$\left.\begin{array}{c}\varphi(\eta_p\lambda_{2p-1,2}\xi_1) \overset{(5.1),(1.2)}{\sim} d_2^{p-1}(a_2b_2^{p-1}x_{2p-1,2}b_2c_2) \overset{(1.5)}{\sim}\\ \overset{(1.5)}{\sim} d_2^{p-1}(b_2^{p-1}x_{2p-1,2}b_2) \overset{(1.2),(5.1)}{\sim} \varphi(\lambda_{2p-1,1}),\\ \varphi(\eta_1\lambda_{2p-1,2}\xi_{p+1}) \overset{(5.1),(1.2)}{\sim} (a_2x_{2p-1,2}d_2^{p-1}c_2)b_2^p \overset{(1.5)}{\sim}\\ \overset{(1.5)}{\sim} d_2^2(x_{2p-1,2}d_2^{p-1})b_2^p \overset{(1.2),(5.1)}{\sim} \varphi(\lambda_{2p-1,1});\end{array}\right\} \quad (4.7)$$

$$\left.\begin{array}{c}\varphi(\eta_{q+1}\lambda_{2q,2}\xi_1) \overset{(5.1),(1.2)}{\sim} d_2^q(a_2b_2^{q-1}x_{2q,2}b_2c_2) \overset{(1.6)}{\sim}\\ \overset{(1.6)}{\sim} d_2^q(b_2^{q-1}x_{2q,2}b_2) \overset{(1.2)}{\sim} d_2x_{2q,2}b_2 \overset{(5.1)}{=} \varphi(\lambda_{2q,1}),\\ \varphi(\eta_1\lambda_{2q,2}\xi_{q+1}) \overset{(5.1),(1.2)}{\sim} (a_2d_2x_{2q,2}d_2^{q-1}c_2)b_2^q \overset{(1.6)}{\sim}\\ \overset{(1.6)}{\sim} (d_2x_{2q,2}d_2^{q-1})b_2^q \overset{(1.2)}{\sim} d_2x_{2q,2}b_2 \overset{(5.1)}{=} \varphi(\lambda_{2q,1});\end{array}\right\} \quad (4.8)$$

$$\left.\begin{array}{c}\varphi(\sigma_1\xi_1) \overset{(1.2)}{\sim} b_1d_2(d_1c_2) \overset{(6.4)}{\sim} b_1(d_2c_0) \overset{(6.4)}{\sim}\\ \overset{(6.4)}{\sim} b_1c_1 \overset{(6.1)}{\sim} (d_2d_0)c_1 \overset{(6.4)}{\sim} a_2b_2 \overset{(5.1)}{=} \varphi(\xi_1),\\ \varphi(\eta_1\sigma_1) \overset{(1.2)}{\sim} a_2(b_2b_1)d_2d_1b_2 \overset{(6.2)}{\sim} (a_2d_0)d_2d_1b_2 \overset{(1.3)}{\sim}\\ \overset{(1.3)}{\sim} (a_1d_2)d_1b_2 \overset{(1.3)}{\sim} (a_0d_1)b_2 \overset{(1.3)}{\sim} \varphi(\eta_1);\end{array}\right\} \quad (4.9)$$

$$\left.\begin{array}{c}\varphi(\sigma_1\sigma_1^{-1}) \overset{(5.1)}{=} (b_1d_2d_1b_2)(d_2b_1b_2d_1) \overset{(1.2)}{\sim} (b_1d_2)(b_2d_1) \overset{(1.2)}{\sim} 1 \overset{(1.2)}{\sim}\\ \overset{(1.2)}{\sim} (d_2b_1)(d_1b_2) \overset{(1.2)}{\sim} (d_2b_1b_2d_1)(b_1d_2d_1b_2) \overset{(5.1)}{=} \varphi(\sigma_1^{-1}\sigma_1);\end{array}\right\} \quad (4.10)$$

$$\left.\begin{array}{c}\varphi(\sigma_2\sigma_1\sigma_2) \overset{(1.2)}{\sim} d_2b_1d_2d_1b_2^2b_1d_2^2d_1b_2^2 \overset{(6.2)}{\sim} d_2(b_1d_2d_1b_2)d_0d_2^2d_1b_2^2 \overset{(6.15)}{\sim}\\ d_2d_0(b_1d_2d_1b_2)d_2^2d_1b_2^2 \overset{(6.1)}{\sim} b_1^2d_2(d_1d_2)d_1b_2^2 \overset{(6.2)}{\sim} b_1^2d_2(d_1d_2)d_1(d_0d_1)b_2 \overset{(6.1)}{\sim}\\ b_1^2d_2b_0d_1(d_0d_1)b_2 \overset{(1.2)}{\sim} b_1^2(d_2b_0d_1d_0b_2)d_2d_1b_2 \overset{(6.21)}{\sim} b_1^2(b_0d_2d_1b_2d_0)d_2d_1b_2\\ \overset{(6.1)}{\sim} b_1^2(d_1d_2)d_2d_1b_2d_0d_2d_1b_2 \overset{(6.2)}{\sim} b_1d_2^2d_1b_2(b_2b_1)d_2d_1b_2 \overset{(1.2)}{\sim} \varphi(\sigma_1\sigma_2\sigma_1);\end{array}\right\} \quad (4.11)$$

$$\left.\begin{aligned}
&\varphi(\lambda_{2p-1,2}\Sigma_{1,p}) \overset{(6.28)}{\sim} (d_2x_{2p-1,2}b_2^2)(b_1d_2^pd_1b_2^p) \overset{(6.2)}{\sim} d_2(x_{2p-1,2}b_2)d_0d_2^pd_1b_2^p\\
&\overset{(6.16)}{\sim} d_2d_0(x_{2p-1,2}b_2)d_2^pd_1b_2^p \overset{(1.2)}{\sim} (d_2d_0)(d_2^{p-1}b_2^{p-1})(x_{2p-1,2}d_2^{p-1})d_1b_2^p\\
&\overset{(6.1)}{\sim} b_1d_2^{p-1}(b_2^{p-1}x_{2p-1,2}d_2^{p-1})d_1b_2^p \overset{(6.18)}{\sim} b_1d_2^{p-1}d_1(b_2^{p-1}x_{2p-1,2}d_2^{p-1})b_2^p\\
&\overset{(1.2)}{\sim} (b_1d_2^{p-1}d_1b_2^{p-1})(x_{2p-1,2}b_2) \overset{(6.28)}{\sim} \varphi(\Sigma_{1,p-1}\lambda_{2p-1,1}),\\
&\varphi(\lambda_{2p-1,1}\bar{\Sigma}_{1,p}) \overset{(6.29)}{\sim} (x_{2p-1,2}b_2)(b_1^pd_2d_1^pb_2) \overset{(1.2)}{\sim}\\
&b_1^{p-1}(d_1^{p-1}x_{2p-1,2}b_2b_1^p)d_2d_1^pb_2 \overset{(6.19)}{\sim} b_1^{p-1}d_2(d_1^{p-1}x_{2p-1,2}b_2b_1^p)d_1^pb_2\\
&\overset{(1.2)}{\sim} (b_1^{p-1}d_2d_1^{p-1}b_2)(d_2x_{2p-1,2}b_2^2) \overset{(6.29),(5.1)}{\sim} \varphi(\bar{\Sigma}_{1,p-1}\lambda_{2p-1,2}),
\end{aligned}\right\} \quad (4.12)$$

$$\left.\begin{aligned}
&\varphi(\lambda_{2q,2}\Sigma_{1,q}) \overset{(6.28)}{\sim} (d_2^2x_{2q,2}b_2^2)(b_1d_2^qd_1b_2^q) \overset{(1.2)}{\sim} d_2^2x_{2q,2}b_2^2b_1d_2^qd_1b_2^q\\
&\overset{(6.2)}{\sim} d_2(d_2x_{2q,2}b_2)d_0d_2^qd_1b_2^q \overset{(6.17)}{\sim} d_2d_0(d_2x_{2q,2}b_2)d_2^qd_1b_2^q \overset{(1.2)}{\sim}\\
&\overset{(1.2)}{\sim} (d_2d_0)(d_2x_{2q,2}d_2^{q-1})d_1b_2^q \overset{(6.1)}{\sim} b_1(d_2x_{2q,2}d_2^{q-1})d_1b_2^q \overset{(2)}{\sim}\\
&\overset{(1.2)}{\sim} b_1d_2^q(b_2^{q-1}x_{2q,2}d_2^{q-1})d_1b_2^q \overset{(6.20)}{\sim} b_1d_2^qd_1(b_2^{q-1}x_{2q,2}d_2^{q-1})b_2^q\\
&\overset{(1.2)}{\sim} (b_1d_2^qd_1b_2^q)(d_2x_{2q,2}b_2) \overset{(6.28),(5.1)}{\sim} \varphi(\Sigma_{1,q}\lambda_{2q,1}),\\
&\varphi(\lambda_{2q,1}\bar{\Sigma}_{1,q}) \overset{(6.29)}{\sim} (d_2x_{2q,2}b_2)(b_1^qd_2d_1^qb_2) \overset{(1.2)}{\sim} d_2x_{2q,2}b_2b_1^qd_2d_1^qb_2\\
&\overset{(1.6)}{\sim} d_2(d_0b_1^{q-2}x_{2q,1}d_1^{q-2}b_0)b_2b_1^qd_2d_1^qb_2 \overset{(6.2),(1.2)}{\sim} b_1^p(d_1x_{2q,1}b_1)d_2d_1^qb_2\\
&\overset{(6.17)}{\sim} b_1^qd_2(d_1x_{2q,1}b_1)d_1^qb_2 \overset{(2)}{\sim} b_1^qd_2d_1(d_1^{q-2}b_0x_{2q,2}d_0b_1^{q-2})d_1^{q-1}b_2\\
&\overset{(6.1),(1.2)}{\sim} (b_1^qd_2d_1^qb_2)(d_2^2x_{2q,2}b_2^2) \overset{(6.29),(5.1)}{\sim} \varphi(\bar{\Sigma}_{1,q}\lambda_{2q,2});
\end{aligned}\right\} \quad (4.13)$$

$$\left.\begin{aligned}
&\varphi(\lambda_{2p-1,1}\Sigma'_{1,p-1}) \overset{(6.30)}{\sim} (x_{2p-1,2}b_2)D_{p,2} \overset{(1.9)}{\sim}\\
&\overset{(1.9)}{\sim} D_{p-1,2}(x_{2p-1,2}b_2) \overset{(6.30)}{\sim} \varphi(\Sigma'_{1,p-2}\lambda_{2p-1,1}),\\
&\varphi(\lambda_{2q,1}\Sigma'_{1,q-1}) \overset{(6.30)}{\sim} (d_2x_{2q,2}b_2)D_{q,2} \overset{(1.10)}{\sim}\\
&\overset{(1.10)}{\sim} D_{q,2}(d_2x_{2q,2}b_2) \overset{(6.30)}{\sim} \varphi(\Sigma'_{1,q-1}\lambda_{2q,1}).
\end{aligned}\right\} \quad (4.14)$$

□

7. Further Approaches to Classification of Spatial Graphs

By Theorems 1.6 and 1.7 the isotopy classification of spatial graphs reduces to a word problem in the semigroups RSG_n, NSG_n. A solution of the word problem will provide an algorithmic classification of spatial graphs up to ambient isotopy.

Problem 7.1. Find an algorithm to decide whether two central elements of the semigroup RSG_n (respectively, NSG_n) are equal.

In Sec. 7.1 the semigroups RSG_n and NSG_n are studied via representation theory of groups. In Lemma 7.6 a presentation for the fundamental group $\pi_1(S^3 - G)$ of a spatial graph G is obtained by means of three-page embeddings. This description will be used to get a lower bound for the three-page complexity $tp(G)$. In Sec. 7.3 a complexity theory for spatial graphs is developed by introducing the three-page complexity $tp(G)$ in Definition 7.10. A lower bound of $tp(G)$ will be obtained in terms of the group $\pi_1(S^3 - G)$, see Proposition 7.20 in Sec. 7.4. In Sec. 7.5 spatial graphs up to complexity 6 are described.

7.1. *Groups associated to the semigroups RSG_n, NSG_n.*

By Theorems 1.6 and 1.7 any representation of RSG_n, NSG_n, that is non-trivial on their centres, gives an isotopy invariant of spatial graphs. Representation theory of groups is simpler than that of semigroups. So, the goal is to map the semigroups RSG_n, NSG_n to groups and then to study representations of these groups.

Definition 7.2 (the group $\widetilde{F}$ associated to a semigroup F). Let F be a finitely presented semigroup generated by a set A and relations R. Put $A^{-1} = \{a^{-1} \mid a \in A\}$. *The associated group* $\widetilde{F}$ is generated by the set $A \cup A^{-1}$ and relations R. The *natural homomorphism* $F \to \widetilde{F}$ is given by $A \to A \cup A^{-1}$.

Lemma 7.3. *The groups $\widetilde{RSG_n}, \widetilde{NSG_n}$ are isomorphic to the free abelian group $\mathbb{Z}^{n+1}$ generated by $\tilde{a}_i, \tilde{x}_m$, where $i \in \mathbb{Z}_3$, $3 \le m \le n$. The natural homomorphisms $RSG_n \to \widetilde{RSG_n}$ and $NSG_n \to \widetilde{NSG_n}$ are defined on the generators*:

$$a_i \mapsto \tilde{a}_i, \quad b_i \mapsto \tilde{a}_{i-1} - \tilde{a}_{i+1}, \quad c_i \mapsto -\tilde{a}_i, \quad d_i \mapsto \tilde{a}_{i+1} - \tilde{a}_{i-1}, \quad x_{2q,i} \mapsto \tilde{x}_{2q},$$
$$x_{2p-1,0} \mapsto \tilde{x}_{2p-1}, \quad x_{2p-1,1} \mapsto \tilde{x}_{2p-1} + \tilde{a}_2 - \tilde{a}_0, \quad x_{2p-1,2} \mapsto \tilde{x}_{2p-1} + \tilde{a}_2 - \tilde{a}_1.$$

Proof. Let $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i, \tilde{d}_i$ be the images of a_i, b_i, c_i, d_i under the natural homomorphisms $RSG_n \to \widetilde{RSG_n}$ and $NSG_n \to \widetilde{NSG_n}$. Relations (1.1)–(1.10) convert to new relations $\widetilde{(1.1)} - \widetilde{(1.10)}$ between words in the letters $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i, \tilde{d}_i$.

Firstly, let us consider the case $n = 2$, i.e. the Dynnikov semigroup $DS = RSG_2 = NSG_2$. The elements $\tilde{b}_i, \tilde{d}_i \in \widetilde{DS}$ are invertible by (1.2). In order to get a presentation of $\widetilde{DS}$, let us add the symbols $\tilde{a}_i^{-1}, \tilde{c}_i^{-1}$ that are inverses of $\tilde{a}_i, \tilde{c}_i$, respectively. We have $\widetilde{(1.3)}$ $\tilde{b}_i = \tilde{a}_{i-1}\tilde{c}_{i+1}, \tilde{d}_i = \tilde{a}_{i+1}\tilde{c}_{i-1}$. Then $\tilde{b}_i = \tilde{a}_{i-1}\tilde{c}_{i+1} = \tilde{c}_{i-1}^{-1}\tilde{a}_{i+1}^{-1}$ or $\tilde{c}_{i-1}\tilde{a}_{i-1}\tilde{c}_{i+1}\tilde{a}_{i+1} = 1$.

Put $\tilde{e}_i = \tilde{c}_i\tilde{a}_i$, $i \in \mathbb{Z}_3$. Hence, $\tilde{e}_2\tilde{e}_1 = \tilde{e}_0\tilde{e}_2 = \tilde{e}_1\tilde{e}_0 = 1$, i.e. $\tilde{e}_0 = \tilde{e}_1 = \tilde{e}_2$ and $\tilde{e}_0^2 = 1$. The relation (1.1) maps to $\widetilde{(1.1)}$ $\tilde{a}_1\tilde{c}_2\tilde{a}_2\tilde{c}_0\tilde{a}_0\tilde{c}_1 = 1$ or $\tilde{c}_1\tilde{a}_1\tilde{c}_2\tilde{a}_2\tilde{c}_0\tilde{a}_0 = 1$, i.e. $\tilde{e}_0^3 = 1$ and $\tilde{e}_0 = \tilde{e}_1 = \tilde{e}_2 = 1$. So, the elements $\tilde{a}_i, \tilde{c}_i$ are mutually inverse.

Then $\widetilde{(1.3)}$ $\tilde{a}_{i+1} = \tilde{a}_{i-1}\tilde{d}_i$ and $\tilde{b}_i = \tilde{a}_{i+1}\tilde{c}_{i+1}$ imply $\tilde{a}_{i+1} = \tilde{a}_{i-1}\tilde{b}_i^{-1}$ and $\tilde{b}_i = \tilde{a}_{i-1}\tilde{a}_{i+1}^{-1}$, respectively. Therefore, $\tilde{a}_{i+1} = \tilde{a}_{i-1}\tilde{a}_{i+1}\tilde{a}_{i-1}^{-1}$, i.e. the elements $\tilde{a}_i$ commute with each other. So, $\widetilde{DS}$ is the free abelian group $\mathbb{Z}^3$ generated by $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2$. The other letters are $\tilde{b}_i = \tilde{a}_{i-1}\tilde{a}_{i+1}^{-1}$, $\tilde{c}_i = \tilde{a}_i^{-1}$, $\tilde{d}_i = \tilde{a}_{i+1}\tilde{a}_{i-1}^{-1}$.

In the general case $n > 2$, it suffices to check that the images $\tilde{x}_{m,i}$ of the letters $x_{m,i}$ commute with each other and with all $\tilde{a}_i$. Recall that commutativity $uv = vu$ is briefly denoted by $u \leftrightarrow v$. Relations

$$\widetilde{(1.5)}\ \tilde{x}_{2p-1,i}\tilde{d}_i^{p-1} = \tilde{a}_i(\tilde{x}_{2p-1,i}\tilde{d}_i^{p-1})\tilde{a}_i^{-1}, \quad \widetilde{(1.6)}\ \tilde{d}_i\tilde{x}_{2q,i}\tilde{d}_i^{q-1} = \tilde{a}_i(\tilde{d}_i\tilde{x}_{2q,i}\tilde{d}_i^{q-1})\tilde{a}_i^{-1}$$

imply $\tilde{x}_{k,i} \leftrightarrow \tilde{a}_i$, where $3 \le k \le n$. By $\widetilde{(1.4)}$ we get $\tilde{a}_i \leftrightarrow \tilde{x}_{k,i\pm1}$. By $\widetilde{(1.8)}$ $\tilde{x}_{2p-1,i}\tilde{b}_i \leftrightarrow \approx x_{m,i+1}$ and $\tilde{d}_i\tilde{x}_{2q,i}\tilde{b}_i \leftrightarrow \tilde{x}_{m,i+1}$ we have $\tilde{x}_{k,i} \leftrightarrow \tilde{x}_{m,i+1}$. By $\widetilde{(1.4)}$ we conclude $\tilde{x}_{m,i+1} \leftrightarrow \tilde{x}_{k,i\pm1}$. The symbols $\tilde{x}_{m,1}, \tilde{x}_{m,2}$ are expressed in terms of $\tilde{x}_m = \tilde{x}_{m,0}$ by $\widetilde{(1.4)}$, $\widetilde{(6.7)}$, $\widetilde{(6.9)}$. ☐

For each $i \in \mathbb{Z}_3$, introduce the linear functions

$$F_i : \widetilde{RSG_n}, \widetilde{NSG_n} \to \mathbb{Z} \quad \text{by } F_i(\tilde{a}_i) = 0, \quad F(\tilde{a}_{i\pm1}) = 1, \quad F_i(\tilde{x}_m) = 0,$$

except $i = 2$ and $m = 2p - 1$. In the latter case, put $F_2(\tilde{x}_{2p-1}) = 1$.

Denote by $|\beta|$ the difference between the number of the left and right brackets in a bracket expression β. The following claim is an easy observation.

Lemma 7.4. *For any $w \in RSG_n, NSG_n$ and $i \in \mathbb{Z}_3$, we have $F_i(\tilde{w}) = |\beta_i(w)|$.*

Now we may describe the images of RSG_n and NSG_n in the associated groups. Elements of the groups $\widetilde{RSG_n}$ and $\widetilde{NSG_n}$ will be written in the abelian form.

Proposition 7.5. *Under the natural homomorphisms $RSG_n \to \widetilde{RSG_n}$ and $NSG_n \to \widetilde{NSG_n}$, the centres of RSG_n, NSG_n map to the set*

$$\left\{- z\tilde{a}_0 - z\tilde{a}_1 + z\tilde{a}_2 + k_3\tilde{x}_3 + \cdots + k_n\tilde{x}_n \,\middle|\, \sum_{2\le p\le \frac{n+1}{2}} k_{2p-1} = 2z,\ k_m \ge 0,\ 3 \le m \le n\right\}.$$

The centre of $DS = RSG_2 = NSG_2$ maps to the zero element $0 \in \widetilde{DS} \cong \mathbb{Z}^3$.

Proof. Let $\tilde{w}$ be the image of a word $w \in RSG_n$ or $w \in NSG_n$ under the natural homomorphism. By Lemma 7.3 the word $\tilde{w} \in \widetilde{RSG_n}, \widetilde{NSG_n}$ has the form $\tilde{w} = x\tilde{a}_0 + y\tilde{a}_1 + z\tilde{a}_2 + \sum_{m=3}^{n} k_m\tilde{x}_m$, where $x, y, z, k_m \in \mathbb{Z}$. Since the words $\beta_i(w)$ are balanced, then $|\beta_i(w)| = 0$ for each $i \in \mathbb{Z}_3$. By Lemma 7.4 we have

$$F_0(\tilde{w}) = y + z = 0, \quad F_1(\tilde{w}) = z + x = 0, \quad F_2(\tilde{w}) = x + y + \sum_p k_{2p-1} = 0,$$

i.e. $x = y = -z$ and $\sum_p k_{2p-1} = 2z$. Conversely, any word $\tilde{w}$ of this type is the image of the central element $w = a_2^z x_{3,0}^{k_3} \cdots x_{n,0}^{k_n} c_0^z c_1^z \in RSG_n, NSG_n$. ☐

The image of the centre of $RSG_3 \cong NSG_3$ is $\{-z\tilde{a}_0 - z\tilde{a}_1 + z\tilde{a}_2 + 2z\tilde{x}_3 \mid z \ge 0\}$. For singular knots, the centre of $RSG_{\{4\}}$ maps to the subset $\{k\tilde{x}_4 \mid k \ge 0\}$ of the group $\mathbb{Z}^4$ generated by $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{x}_4$. If the image of a word in $\widetilde{RSG_{\{4\}}}$ is $\widetilde{w_G} = k\tilde{x}_4$, then the given singular knot G contains exactly k singular points.

More generally, only the set of the vertices of a spatial graph G can be reconstructed from the image of an encoding word w_G in the associated group. It means that the algebrac approach to the method of three-page embeddings can be effective only via semigroups, not via groups.

7.2. *Presentation of the fundamental group* $\pi_1(S^3 - G)$

Further spatial graphs $G \subset S^3$ are considered up to homeomorphism $f : S^3 \to S^3$, which can reverse the orientation of S^3. Assume that S^3 is obtained from $\mathbb{R}^3$ by adding the infinity point ∞. For a knot (or a graph) $K \subset S^3$, the fundamental group $\pi_1(S^3 - K)$ is said to be *the knot group* $\pi(K)$ or *the graph group*.

Neuwirth constructed a presentation of the knot group by using an arc presentation of a given knot [13]. *The arc presentation* is an embedding of a knot into a book with finitely many pages in such a way that each page contains exactly one arc. The Neuwirth construction will be modified for three-page embeddings.

Adding the infinity point ∞ to the axis α gives the circle $\bar{\alpha} \subset S^3$. Let us demonstrate our computations for the trefoil K in Fig. 15. We are going to choose *the Neuwirth loops* lying in S^3 near the pages P_i, the base point is $\infty \in \bar{\alpha}$. For each arc $\gamma \subset K \cap P_i$, let us take a loop going around γ and all the arcs lying in P_i farther from α than γ. See the right picture of Fig. 15.

For example, for the arc $A_1A_2 \subset P_1$ in Fig. 15, we take the loop r near P_1. Similarly, the arc $A_1A_3 \subset P_0$ provides the loop u_0 near P_0. The chosen loops will be *the generators* of $\pi(K)$. To each segment $A_jA_{j+1} \subset \alpha$ associate two or three loops (at most one near each page) going around nearest arcs. To get defining *Neuwirth's relations* let us write down the associated loops from P_0 to P_2. The segment $A_1A_2 \subset \alpha$ provides $u_0 r = 1$, the segment A_2A_3 gives $u_0 v_1 = 1$.

In fact, we get the following presentation:

$$\pi(K) = \langle u_0, u_1, u_2, u_3, r, s, t, v_1, v_2, v_3 \mid u_0 r = u_0 v_1 = s v_1 = u_1 s v_2$$
$$= u_2 s v_3 = u_3 v_3 = u_3 t^{-1} v_2 = u_2 t^{-1} v_1 = u_1 t^{-1} = 1\rangle.$$

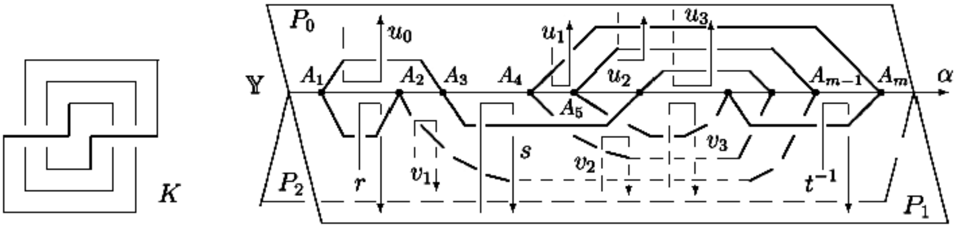

Fig. 15. The trefoil K is encoded by the word $w_K = a_2 d_0 d_2 a_1^2 b_2 b_0 c_1^2 c_2$.

We have $r^{-1} = v_1^{-1} = u_0 = u_2 t^{-1} = s$, $v_2^{-1} = u_1 s = u_3 t^{-1}$, $v_3^{-1} = u_2 s = u_3$, $u_1 = t$, hence $u_0 = s$, $u_1 = t$, $u_2 = st$, $u_3 = tst$. Relation $u_2 s = u_3$ converts to $sts = tst$. We get the presentation $\pi(K) = \langle s, t \mid sts = tst \rangle$ of the trefoil group.

Lemma 7.6. *The above modification of the Neuwirth construction provides a presentation of the graph group* $\pi(G) = \pi_1(S^3 - G)$ *for any spatial graph* $G \subset \mathbb{Y}$.

Proof. For each segment $A_j A_{j+1} \subset \alpha$, choose a small subsegment $I_j \subset A_j A_{j+1}$. For sufficiently small $\varepsilon > 0$, let $N(I_j)$ be the ε-neighborhood (a cylinder) of I_j, and $N(A_1 A_m)$ be the $\frac{\varepsilon}{2}$-neighborhood (also a cylinder) of the segment $A_1 A_m$. We join the two neighboring cylinders $N(I_j), N(I_{j+1})$ by an arc $T_j \subset \mathbb{R}^3 - \mathbb{Y}$. Put

$$X = N(I_1) \cup T_1 \cup N(I_2) \cup \cdots \cup T_{m-2} \cup N(I_{m-1}), \quad Y = S^3 - (G \cup N(A_1 A_m)).$$

Then the space X is contractible, i.e. $\pi_1(X) = 1$. The group $\pi_1(Y)$ is freely generated by the Neuwirth loops. Moreover, the space $X \cup Y$ is homotopically equivalent to $S^3 - G$, hence $\pi_1(X \cup Y) = \pi(G)$.

By Seifert–Van–Kampen's theorem, in order to get a presentation of $\pi_1(X \cup Y)$ we should add relations corresponding to all generators of $\pi_1(X \cap Y)$. The intersection $X \cap Y$ consists of the tubes $N(I_j) - N(A_1 A_m)$ joined by the arcs T_j. The group $\pi_1(X \cap Y)$ is generated by loops going around the segments $I_j \subset A_j A_{j+1}$. All defining relations of $\pi(G)$ are Neuwirth's relations introduced above. □

Definition 7.7 (the disjoint union, a vertex sum, an edge sum, a loop sum).

(a) A spatial graph $F \subset S^3$ is called *the disjoint union* of spatial graphs $G, H \subset S^3$ and denoted by $G \sqcup H$, if there is a two-sided 2-sphere $S \subset S^3$ such that $F = G \cup H$, the subgraph $G \subset F$ lies inside the sphere S, the subgraph $H \subset F$ lies outside S.

(b) A spatial graph $F \subset S^3$ is called *a vertex sum* of spatial graphs $G, H \subset S^3$ and denoted by $G * H$, if there is a two-sided 2-sphere $S \subset S^3$ such that $F = G \cup H$, $F \cap S = v$ is either a vertex or a point inside a loop of G and H, the subgraph $G - v$ lies inside the sphere S, $H - v$ lies outside S, see Fig. 16.

(c) A spatial graph $F \subset S^3$ is called *an edge sum* of spatial graphs $G, H \subset S^3$ and denoted by $G \vee H$, if there is a two-sided 2-sphere $S \subset S^3$ and an edge $e \subset F$ such that $F - e = G \cup H$, $F \cap S = e \cap S = 1$ point, G lies inside S, H lies outside S.

(d) A spatial graph $F \subset S^3$ is called *a loop sum* of spatial graphs $G, H \subset S^3$ and denoted by $G \circ H$, if there is a two-sided 2-sphere $S \subset S^3$ and an arc $I \subset S$ such that $F = (G \cup H) - I$, $G \cap H = I$ is contained in a loop $e_G \subset G$ and in a loop $e_H \subset H$, the subgraph $G - I \subset F$ lies inside S, $H - I$ lies outside S, see Fig. 16.

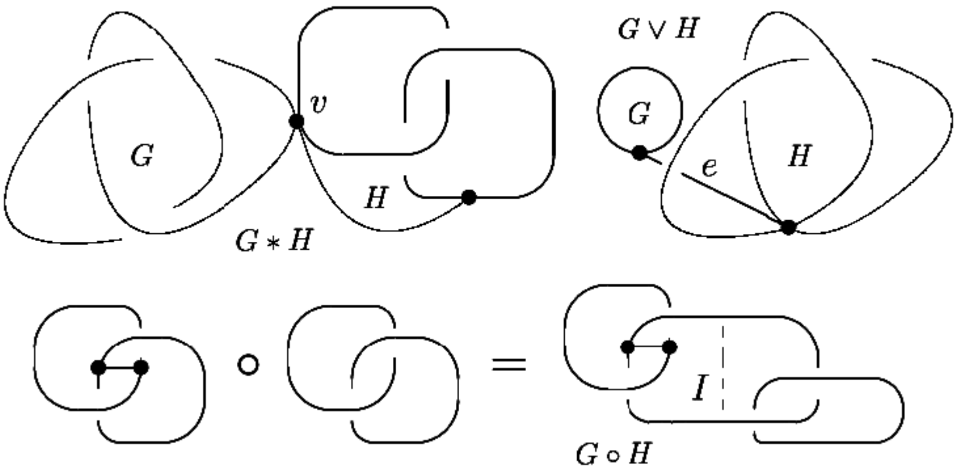

Fig. 16. A vertex sum of spatial graphs, an edge sum, a loop sum.

The disjoint union $G \sqcup H$ is determined uniquely up to homeomorphism $f : S^3 \to S^3$. In the case of links, the notion of a loop sum coincides with the usual connected sum. For arbitrary spatial graphs, a loop sum can be nonassociative and noncommutative since it depends on the loops e_G, e_H from Definition 7.7(d).

For finitely presented groups π, π', the group $\pi * \pi'$ is called *the free product* of π, π'. A presentation of $\pi * \pi'$ can be obtained by uniting the generators and defining relations of π, π'. Lemma 7.6 implies

Lemma 7.8. *For any spatial graphs* $G, H \subset S^3$, *we have*

$$\pi(G \sqcup H) \cong \pi(G * H) \cong \pi(G \vee H) \cong \pi(G) * \pi(H).$$

Due to Lemma 7.8 we are able to calculate the Alexander polynomial of spatial graphs by means of three-page embeddings. This allows us to classify an infinite family of singular knots with arbitrary numbers of singular points and crossings [9, Propositions 2.4–2.5].

7.3. *Complexity theory for spatial graphs*

It is convenient to extend a three-page embedding to a more general one.

Definition 7.9 (general three-page embeddings). An embedding $G \subset \mathbb{Y}$ of a graph is called *a general three-page embedding*, if conditions (a)–(c) of Definition 3.1 hold and

(f) a neighborhood of each m-vertex $A \in G$ lies in $P_i \cup P_j \subset \mathbb{Y}$.

A general three-page embedding of a spatial graph can be constructed as in Sec. 3.2, but there is no need to check conditions (iv)–(vi) there. General three-page embeddings also can be encoded by finitely many letters. We may prove analogues of Theorems 1.6 and 1.7, but in this general case universal semigroups will be generated by much more letters and defining relations.

Definition 7.10 (the arch number $ar(G)$ and three-page complexity $\mathrm{tp}(G)$).

(a) *An arch* of a general three-page embedding $G \subset \mathbb{Y}$ is a connected component of $G - \alpha$. *The arch number* $\mathrm{ar}(G)$ is the number of arches in the embedding.
(b) *The three-page complexity* $\mathrm{tp}(G)$ is the minimum of $\mathrm{ar}(G) - 2$ over all possible general three-page embeddings $G \subset \mathbb{Y}$ of a given spatial graph G.

In the case of a link $L \subset \mathbb{Y}$, the arch number $\mathrm{ar}(L)$ is equal to the length of w_L. In general, we have $\mathrm{tp}(G) \geq 0$ for a spatial graph G. Moreover, $\mathrm{tp}(G) = 0$ if and only if G is the unknot O_1. For the Hopf link L, we have $\mathrm{tp}(L) = 4$.

Let $S(p,q)$ be the nonoriented 2-bridge link having Shubert's normal form with the parameters $p, q \geq 1$ [2, Chap. 12.A]. The link $S(p,q)$ can be encoded by

$$a_0 a_1^{p-1} b_2 b_1^{q-1} b_0 c_1^{p-q} d_1^{q-1} c_2 c_1^{q-1}, \quad \text{hence } \mathrm{tp}(S(p,q)) \leq 2p + 2q - 2.$$

Conjecture 7.11. The three-page complexity of $S(p,q)$ is $2p+2q-2$, $p+q \geq 3$.

Conjecture 7.11 is true for the Hopf link $S(2,1)$ and trefoil $S(3,1)$.

Lemma 7.12. *For any $k \in \mathbb{N}$, there is a finite number of spatial graphs G of three-page complexity* $\mathrm{tp}(G) = k$.

Proof. Let us estimate from above the number TP_k of all three-page embeddings $G \subset \mathbb{Y}$ of spatial graphs with $\mathrm{ar}(G) = k$. For such a three-page embedding, the number of the intersection points from $G \cap \alpha$ is not more than k. A neighborhood of an m-vertex can be embedded into two pages of $\mathbb{Y}$ in not more than 4^m different monotone ways. We may estimate TP_k very roughly as $TP_k \leq (4^k)^k$. □

A spatial graph $G \subset S^3$ is called *prime*, if it is not a loop sum of other spatial graphs disctinct from the unknot O_1.

Problem 7.13. Find asymptotics for the number $N_n(k)$ of all prime spatial n-graphs of three-page complexity k.

To get additivity (a) of Theorem 1.8 we need a geometric inversion.

Definition 7.14 (the geometric inversion $f_{a,r} : S^3 \to S^3$). Let $S_{a,r} \subset S^3$ be the geometric 2-sphere with a centre $a \in S^3$ and a radius $r > 0$. Then the geometric inversion $f_{a,r} : S^3 \to S^3$ is defined by $f_{a,r}(x) = a + \frac{r^2}{|x-a|^2}(x-a)$. Here $x, a \in S^3$ are considered as usual 3-dimensional vectors, $|x-a|$ is the length of the vector $x-a$. In particular, $f_{a,r}(a) = \infty$, $f_{a,r}(\infty) = a$, $f_{a,r}(x) = x$ for each $x \in S_{a,r}$. Moreover, the inversion changes the orientation of S^3.

If the centre a of a geometric inversion $f_{a,r}$ lies in the axis α of the book $\mathbb{Y}$, then $f_{a,r}(\alpha) = \alpha$ and $f_{a,r}(\mathbb{Y}) = \mathbb{Y}$. Assume that the arches of a spatial graph $G \subset \mathbb{Y}$ come to the axis α as perpendicular smooth curves. Any three-page embedding of a graph is isotopic inside $\mathbb{Y}$ to such *a geometric embedding*.

For a geometric three-page embedding $K \subset \mathbb{Y}$ of a singular knot K, two branches of K intersect non-transversally at a singular point $A \in K$, see the singular knot 2_2 in Fig. 17. But we may always select the branches of K at A since any rigid isotopy keep a neighborhood of A in a (non-constant) plane or a bowed disk.

Under an inversion $f_{a,r}$, $a \in \alpha$, a smooth arch with endpoints $b, c \in \alpha$ goes to a smooth arch with endpoints $f_{a,r}(b), f_{a,r}(c) \in \alpha$, in the same page.

Lemma 7.15. *Let e_G be a loop of a spatial graph G. Let $G \subset \mathbb{Y}$ be a general three-page embedding. Then there is a geometric inversion $f_{a,r}$ such that $f_{a,r}(G) \subset \mathbb{Y}$, $\mathrm{ar}(f_{a,r}(G)) = \mathrm{ar}(G)$, the left extreme point from $f_{a,r}(G) \cap \alpha$ belongs to $f_{a,r}(e_G)$.*

Proof. The loop e_G has two extreme points from $e_G \cap \alpha$, say A_k, A_l, where $k < l$. Since the loop e_G contains not more than one vertex of G, then one of these points (say A_k) is not a vertex of G. Both geometric arches at A_k come perpendicularly to the axis α. Then we may take a geometric sphere $S_{a,r}$ with $a \in \alpha$ and a small radius r, such that $A_k \in S_{a,r}$ and $G - A_k$ lies outside $S_{a,r}$.

Then $f_{a,r}(A_k) = A_k$, and $f_{a,r}(G - A_k)$ lies inside $S_{a,r}$. The point A_k is now an extreme point of $f_{a,r}(G) \cap \alpha$, in the axis α. If A_k is the right extreme point, then take a symmetric reflection of $f_{a,r}(G) \subset \mathbb{Y}$ in a plane perpendicular to α. □

Proposition 7.16. *For any spatial graphs $G, H \subset S^3$, we have*

$$\text{(a)}\quad \mathrm{tp}(G \sqcup H) = \mathrm{tp}(G) + \mathrm{tp}(H) + 2;$$
$$\text{(b)}\quad \mathrm{tp}(G * H) = \mathrm{tp}(G) + \mathrm{tp}(H) + 2;$$
$$\text{(c)}\quad \mathrm{tp}(G \vee H) = \mathrm{tp}(G) + \mathrm{tp}(H) + 3;$$
$$\text{(d)}\quad \mathrm{tp}(G \circ H) \le \mathrm{tp}(G) + \mathrm{tp}(H).$$

Proof. (a) Take general three-page embeddings $G, H \subset \mathbb{Y}$ with the minimal numbers of arches, i.e. $\mathrm{ar}(G) = \mathrm{tp}(G)+2$, $\mathrm{ar}(H) = \mathrm{tp}(H)+2$. To get a general three-page embedding $G \sqcup H \subset \mathbb{Y}$, we attach two copies of $\mathbb{Y}$ along α. Then

$$\mathrm{tp}(G \sqcup H) \le \mathrm{ar}(G \sqcup H) - 2 = \mathrm{ar}(G) + \mathrm{ar}(H) - 2 = \mathrm{tp}(H) + \mathrm{tp}(H) + 2.$$

Conversely, choose a general three-page embedding $G \sqcup H \subset \mathbb{Y}$ such that $\mathrm{ar}(G \sqcup H) = \mathrm{tp}(G \sqcup H) + 2$. The sphere S from Definition 7.7(a) divides the embedding $G \sqcup H \subset \mathbb{Y}$ into two disjoint parts in such a way that $w_{G \sqcup H} = u_1 v_1 \cdots u_k v_k$, where the words $u_1 \cdots u_k$ and $v_1 \cdots v_k$ encode the subgraphs $G, H \subset \mathbb{Y}$. So, we have $\mathrm{ar}(G) + \mathrm{ar}(H) = \mathrm{ar}(G \sqcup H) = \mathrm{tp}(G \sqcup H) + 2$. We get the desired inequality

$$\mathrm{tp}(G) + \mathrm{tp}(H) \le \mathrm{ar}(G) + \mathrm{ar}(H) - 4 = \mathrm{tp}(G \sqcup H) - 2.$$

(b) Take general three-page embeddings $G, H \subset \mathbb{Y}$ such that $\mathrm{ar}(G) = \mathrm{tp}(G) + 2$, $\mathrm{ar}(H) = \mathrm{tp}(H) + 2$. Choose general three-page embeddings $G, H \subset \mathbb{Y}$ such that $\mathrm{ar}(G) = \mathrm{tp}(G) + 2$, $\mathrm{ar}(H) = \mathrm{tp}(H) + 2$, and also the right extreme point of $G \cap \alpha$ (respectively, the left extreme point of $H \cap \alpha$) is the gluing point v from Definition 7.7(b). Attach the obtained embeddings $G, H \subset \mathbb{Y}$ to get a general three-page embedding $G * H \subset \mathbb{Y}$ such that $\mathrm{ar}(G * H) = \mathrm{ar}(G) + \mathrm{ar}(H)$ as (a).

Conversely, take a general embedding $G*H \subset \mathbb{Y}$ with $\mathrm{ar}(G*H) = \mathrm{tp}(G \sqcup H)+2$. The sphere S from Definition 7.7(b) divides the embedding $G*H \subset \mathbb{Y}$ into two parts intersecting at the point v. These parts form two independent general embeddings $G, H \subset \mathbb{Y}$ with $\mathrm{ar}(G) + \mathrm{ar}(H) = \mathrm{ar}(G * H)$. The proof finishes as in (a).

The item (c) is similar to (a) and (b). Given general three-page embeddings $G, H \subset \mathbb{Y}$, by Lemma 7.15 we may construct a general three-page embedding $G \sqcup H \subset \mathbb{Y}$ such that the right extreme point of $G \cap \alpha$ (respectively, the left extreme point of $H \cap \alpha$) is an endpoint of the edge $e \subset G \vee H$ from Definition 7.7(c).

Also we may assume that neighborhoods of these endpoints lie in two common pages of $\mathbb{Y}$. Otherwise it suffices to rotate the embedding of G (say) to secure the above condition. Now we are able to add the edge $e \subset G \vee H$ to the embedding $G \sqcup H \subset \mathbb{Y}$ and to get a general three-page embedding $G \vee H \subset \mathbb{Y}$ with $\mathrm{ar}(G \vee H) = \mathrm{ar}(G) + \mathrm{ar}(H) + 1$. The proof finishes as in the item (b).

The item (d) is similar to the first part of (c). Take general three-page embeddings $G, H \subset \mathbb{Y}$ with $\mathrm{ar}(G) = \mathrm{tp}(G) + 2$, $\mathrm{ar}(H) = \mathrm{tp}(H) + 2$. By Lemma 7.15 we may construct general three-page embeddings $G, H \subset \mathbb{Y}$ that are intersected at a common "vertical" arc $I \perp \alpha$ from Definition 7.7(d), see Fig. 16. Here monotone condition (e) of Definition 2.2 does not play any role. Now the arc I can be removed from the union $G \cup H \subset \mathbb{Y}$ to get a general three-page embedding $G \circ H \subset \mathbb{Y}$ such that $\mathrm{ar}(G \circ H) = \mathrm{ar}(G) + \mathrm{ar}(H) - 2 = \mathrm{tp}(G) + \mathrm{tp}(H) + 2$. Then

$$\mathrm{tp}(G \circ H) \leq \mathrm{ar}(G \sqcup H) - 2 = \mathrm{ar}(G) + \mathrm{ar}(H) - 4 = \mathrm{tp}(H) + \mathrm{tp}(H). \qquad \square$$

The reverse of the item (d) is much harder since the sphere S from Definition 7.7(d) may intersect an embedding $G \circ H \subset \mathbb{Y}$ in a terrible way.

Conjecture 7.17. The three-page complexity is additive under a loop sum, i.e. $\mathrm{tp}(G \circ H) = \mathrm{tp}(G) + \mathrm{tp}(H)$ for any spatial graphs G, H.

If Conjecture 7.17 is true, then we get an hierarchy on the set of spatial graphs considered up to homeomorphism $f : S^3 \to S^3$. Proposition 7.16(a) implies that the three-page complexity of trivial k-component link O_k is $\mathrm{tp}(O_k) = k \cdot \mathrm{tp}(O_1) + 2(k-1) = 2k - 2$. Theorem 1.8 follows from Lemma 7.12 and Proposition 7.16.

7.4. *Lower bound of the three-page complexity*

The crucial problem in a complexity theory is to find a sharp lower bound for the complexity. This will be done for the three-page complexity $\mathrm{tp}(G)$ in terms of the group $\pi(G) = \pi_1(S^3 - G)$.

Definition 7.18 (the three-letters complexity tl(π) of a group).

(a) Let π be a finitely presented group. A presentation of π is called *a three-letters presentation*, if it contains k generators and not more than $k-1$ relations, each relation consists of 3 generators or their inverses.
(b) *The three-letters complexity* $\mathrm{tl}(\pi)$ is the minimal number of generators over all three-letters presentations of π if they exist, otherwise put $\mathrm{tl}(\pi) = \infty$.

For instance, $\mathbb{Z}$ is the only group of tl $= 1$. The groups of tl $= 2$ are $\mathbb{Z} * \mathbb{Z}$ and $\mathbb{Z}_3 * \mathbb{Z}$. The group $\mathbb{Z} \oplus \mathbb{Z} \cong \langle a, b, c \mid abc = acb = 1 \rangle$ has tl $= 3$. All groups of tl $= 3$ are listed in [9, Example 2.11]. The cyclic groups $\mathbb{Z}_k$ $(k > 1)$ have tl $= \infty$.

Proposition 7.19.

(a) *There are finitely many groups* π *of* $\mathrm{tl}(\pi) = k$ *for fixed* k.
(b) *If* π_1 *and* π_2 *have three-letters presentations, then* $\mathrm{tl}(\pi_1 * \pi_2) = \mathrm{tl}(\pi_1) + \mathrm{tl}(\pi_2)$.

Proof.

(a) It suffices to estimate from above the number TL_k of all three-letters presentations of complexity k. For such a presentation, there are not more than 3^k different relations, hence $TL_k \leq (3^k)^{k-1}$.
(b) Since the union of three-letters presentations for π_1, π_2 gives a three-letters presentation for $\pi_1 * \pi_2$, then $\mathrm{tl}(\pi_1 * \pi_2) \leq \mathrm{tl}(\pi_1) + \mathrm{tl}(\pi_2)$. If a defining relation from a three-letters presentation of $\pi_1 * \pi_2$ contains two generators of π_1 (say) and a generator g of π_2, then $g \in \pi_1$ that is a contradiction.

So, any three-letters presentation of $\pi_1 * \pi_2$ splits into two three-letters presentation for π_1 and π_2. Hence, one gets $\mathrm{tl}(\pi_1 * \pi_2) \geq \mathrm{tl}(\pi_1) + \mathrm{tl}(\pi_2)$. □

The θ_k-graph consists of 2 vertices joined by $k \geq 2$ edges. *The trivial graph* $\theta_k \subset S^3$ is the θ_k-graph embedded into $\mathbb{R}^2 \subset S^3$. For example, the trivial graph θ_2 is the unknot. The trivial graphs θ_3 and θ_4 are the second and fourth graphs in Fig. 17 below. Each trivial graph θ_k has a general three-page embedding $\theta_k \subset \mathbb{Y}$ such that $\mathrm{ar}(\theta_k) = k$ and $\theta_k \cap \alpha = 2$ points.

Proposition 7.20. *For any spatial graph* G, *distinct from a trivial graph* θ_k, *we have* $\mathrm{tp}(G) \geq \mathrm{tl}(\pi(G))$. *For the trivial graph* θ_k, *we get* $\pi(\theta_k) = F_{k-1}$, *the free group with* $k-1$ *generators, and* $\mathrm{tl}(\pi(\theta_k)) = k-1$, $\mathrm{tp}(\theta_k) = k-2$.

Table 1. The number of spatial graphs up to three-page complexity 6.

Spatial graphs	tp = 0	tp = 1	tp = 2	tp = 3	tp = 4	tp = 5	tp = 6
Nonoriented knots	1	0	0	0	0	0	1
Nonoriented links	0	0	0	0	1	0	0
Nonoriented spatial 3-graphs	0	1	0	1	2	2	2
Nonoriented singular knots	0	0	2	0	2	2	5

Proof. Let us take a general three-page embedding $G \subset \mathbb{Y}$ with the minimal number of arches, i.e. $\mathrm{ar}(G) = \mathrm{tp}(G) + 2$. Lemma 7.6 gives a presentation of $\pi(G)$ with $\mathrm{ar}(G)$ generators, all relations contain at most three letters.

Two Neuwirth's relations corresponding to the extreme segments $A_1A_2, A_{l-1}A_l$ contain exactly two letters. Hence at least two generators are superfluous, i.e. $tl(\pi(G)) \leq \mathrm{ar}(G) - 2 = \mathrm{tp}(G)$. The above argument does not work, if there is exactly one extreme segment, i.e. $A_1A_2 = A_{l-1}A_l$ and $G = \theta_k$. $\square$

Problem 7.21. Find lower bounds for the three-page complexity in terms of known polynomial invariants of links and spatial graphs.

7.5. *Spatial graphs up to complexity 6*

Figures 17–20 show all nonoriented links, spatial 3-graphs and singular knots of three-page complexity ≤ 6, except disjoint unions. Figures 17–20 contain only two non-trivial links: Hopf link 4_1 and trefoil 6_1.

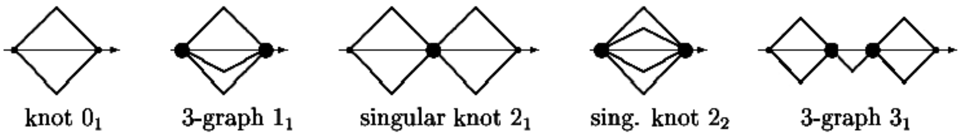

Fig. 17. Spatial graphs up to complexity 3.

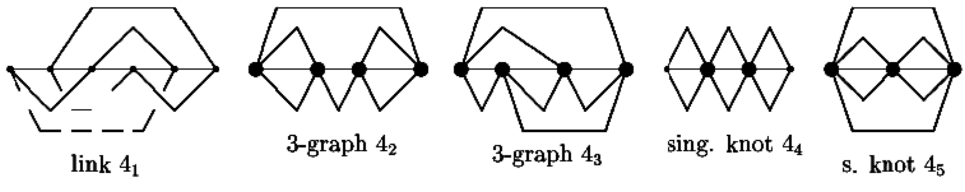

Fig. 18. Spatial graphs of complexity 4.

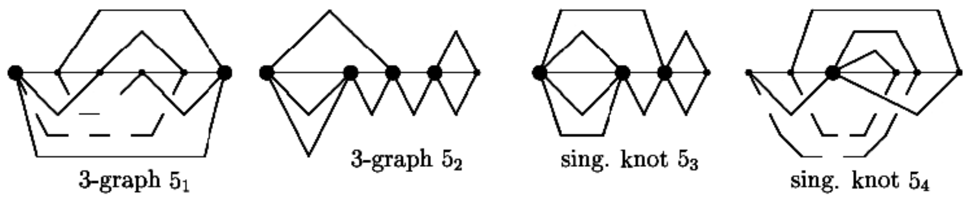

Fig. 19. Spatial graphs of complexity 5.

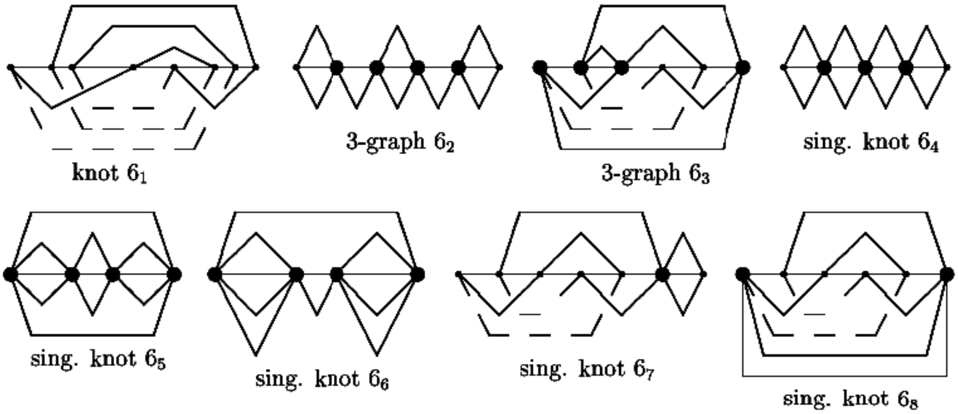

Fig. 20. Spatial graphs of complexity 6.

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