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Three-Page Embeddings of Singular Knots*

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Abstract. A semigroup with 15 generators and 84 relations is constructed. The center of the semigroup is in a one-to-one correspondence with the set of all isotopy classes of nonoriented singular knots (links with finitely many double intersections in general position) in $\mathbb{R}^3$.

Key words: isotopy classification, singular knot, three-page embedding, universal semigroup, knotted graph.

1. Introduction

1.1. Statement of the problem and the results. We develop the Dynnikov method of three-page embeddings for *links with singularities* of the following type: finitely many double intersections in general position are possible. More precisely, the isotopy classification problem is solved for nonoriented singular knots in $\mathbb{R}^3$. The key idea is to construct suitable three-page embeddings for neighborhoods of singular points in a given singular knot. In particular, this construction makes it possible to reduce the number of generators and defining relations in the universal semigroup for singular knots.

1.2. Review of the previous results. An embedding of a link in a structure resembling an open book with finitely many pages was probably for the first time considered by Brunn in 1898 [4]. Namely, he proved that each knot is isotopic to a knot that is projected to the plane $\mathbb{R}^2$ with only one singular point. Later such embeddings were studied in [5–7] and were used in [3]. These investigations provided a new link invariant, the *arc index* [5, 7, 19]. It turned out that each link can be embedded in a book with only 3 pages. In 1999 Dynnikov classified all nonoriented links in $\mathbb{R}^3$ up to ambient isotopy encoding them by *three-page diagrams* [9, 10]. More precisely, we call these diagrams *three-page embeddings* (for the definition, see Sec. 2.1). Dynnikov constructed a semigroup whose center is in a one-to-one correspondence with the set of all isotopy classes of nonoriented (regular) links in $\mathbb{R}^3$. Applying embeddings in a book with arbitrary many pages, Dynnikov reduced the number of relations in his semigroup [11]. Similarly, the first author established an isotopic classification of nonoriented knotted 3-valent graphs in $\mathbb{R}^3$ [18].

1.3. Motivation. Singular knots were called *chimerical graphs* in [15] and four-valent graphs with *rigid* vertices in [16]. The study of singular knots and braids was motivated by the theory of Vassiliev invariants [2]. The corresponding algebraic object in braids is called the *Baez–Birman monoid* or *singular braid monoid* [1, 2]. Some of its properties were investigated in [8, 12, 14]. For singular braids, an analog of Markov's theorem was proved [13]. Many invariants of regular (nonsingular) links, in particular, the Alexander–Conway and Jones polynomials and Vassiliev invariants were constructed for singular knots [15–17, 20]. Homological properties of singular braids on infinitely many strings were studied by the second author [22, 23].

1.4. Main definitions. We work in the PL-category, i.e., the images of circles under an immersion in $\mathbb{R}^3$ are finite polygonal lines. Formally, a *singular knot* is an immersion of several circles in $\mathbb{R}^3$ with possible double intersections in general position at finitely many *singular points* (Figs. 1 and 2). Two *branches* of a given singular knot pass through each singular point. In the present paper (except Sec. 3.5), we consider only nonoriented singular knots, possibly disconnected.

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Note that singular knots are 4-valent graphs embedded in $\mathbb{R}^3$. An *ambient PL-isotopy* between two graphs is a continuous family of PL-homeomorphisms $\phi_t\colon \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,1]$, such that $\phi_0 = \mathrm{id}$ and ϕ_1 sends one graph to another. Singular knots are considered up to rigid ambient PL-isotopy. By definition, any rigid isotopy preserves the rigidity (or *template* by the terminology in [15]) of each singular point. If we allow also nonrigid isotopies, then we arrive at the notion of a *knuckle* 4-valent graph (a graph with *nonrigid* vertices according to the terminology in [16]). In contrast to the case of singular knots, an isotopy of knuckle graphs can arbitrarily permute the edges at each vertex. In Sec. 3.5, we state classification Theorem 5 for knuckle 4-valent graphs. By analogy with regular links, one can represent singular knots by plane diagrams modulo the Reidemeister moves $R1$–$R5$ (see Fig. 1) [15]. We present only PL-analog of the corresponding smooth moves omitting subdivisions and edge breaks. In the case of knuckle 4-valent graphs, the move $R5'$ is taken instead of $R5$.

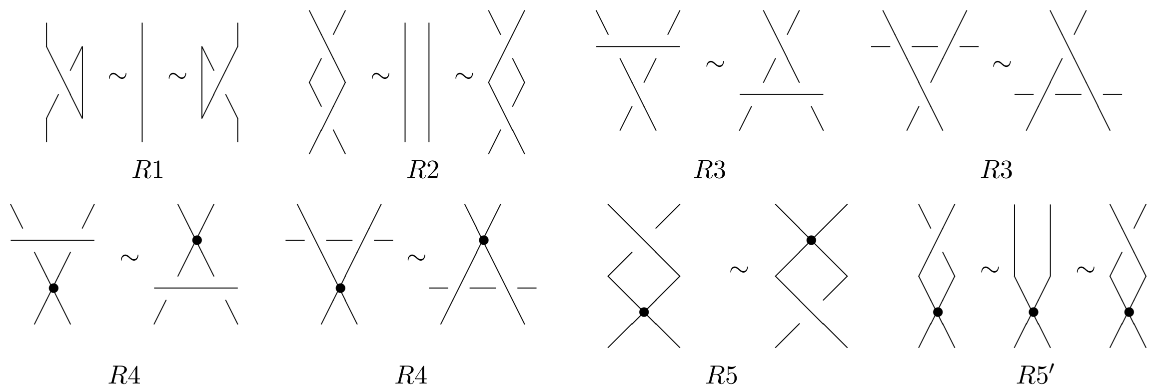

Fig. 1. Reidemeister moves for singular knots and knuckle graphs

1.5. The universal semigroup for singular knots. Everywhere below, the index i belongs to the group $\mathbb{Z}_3 = \{0,1,2\}$. Consider the alphabet $\mathbb{A} = \{a_i, b_i, c_i, d_i, x_i,\ i \in \mathbb{Z}_3\}$ with 15 letters. (For their geometrical interpretation, see Fig. 3.) Let SK be the semigroup with 15 generators in the alphabet $\mathbb{A}$ and the following relations (1)–(10), which correspond to some "elementary ambient isotopies" of singular knots in $\mathbb{R}^3$:

$$a_i = a_{i+1}d_{i-1}, \quad b_i = a_{i-1}c_{i+1}, \quad c_i = b_{i-1}c_{i+1}, \quad d_i = a_{i+1}c_{i-1}, \tag{1}$$

$$x_i = d_{i+1}x_{i-1}b_{i+1}, \tag{2}$$

$$d_0d_1d_2 = 1, \tag{3}$$

$$b_id_i = d_ib_i = 1, \tag{4}$$

$$d_ix_id_i = a_i(d_ix_id_i)c_i, \quad b_ix_ib_i = a_i(b_ix_ib_i)c_i, \tag{5}$$

$$x_i(d_{i+1}d_id_{i-1}) = (d_{i+1}d_id_{i-1})x_i, \tag{6}$$

$$(d_ic_i)w = w(d_ic_i), \quad \text{where } w \in \{c_{i+1}, x_{i+1}, b_id_{i+1}d_i\}, \tag{7}$$

$$(a_ib_i)w = w(a_ib_i), \quad \text{where } w \in \{a_{i+1}, b_{i+1}, c_{i+1}, x_{i+1}, b_id_{i+1}d_i\}, \tag{8}$$

$$t_iw = wt_i, \quad \text{where } t_i = b_{i+1}d_{i-1}d_{i+1}b_{i-1},\ w \in \{a_i, b_i, c_i, x_i, b_{i-1}d_id_{i-1}\}, \tag{9}$$

$$(d_ix_ib_i)w = w(d_ix_ib_i), \quad \text{where } w \in \{a_{i+1}, b_{i+1}, c_{i+1}, x_{i+1}, b_id_{i+1}d_i\}. \tag{10}$$

One of the relations in (4) is superfluous, namely, it can be obtained from (3) and the remaining relations in (4). Hence the total number of relations in (1)–(10) is 84.

1.6. Algebraic classification of singular knots.

Theorem 1. *Each singular knot can be represented by an element of the semigroup SK.*

Theorem 2. *Two singular knots are ambiently isotopic in $\mathbb{R}^3$ if and only if the corresponding elements of the semigroup SK are equal.*

Theorem 3. *An arbitrary element of the semigroup SK corresponds to a singular knot if and only if this element is central, i.e., it commutes with every element of SK.*

As will be shown in Theorem 4 the whole semigroup SK describes a wider class of *three-page singular tangles*. The subsemigroup in SK that is generated by 12 letters a_i, b_i, c_i, d_i ($i \in \mathbb{Z}_3$) and 48 relations in (1)–(10) containing no letters x_i ($i \in \mathbb{Z}_3$) coincides with Dynnikov's semigroup DS in [9, 10]. The center of the semigroup DS classifies all nonoriented (regular) links in $\mathbb{R}^3$ up to ambient isotopy.

1.7. The content of the paper. In Sec. 2.1, we define three-page embeddings of singular knots. These embeddings are constructed from plane diagrams of knots in Sec. 2.2 and are encoded in Sec. 2.3. Theorem 1 is proved in Sec. 2.5. The ordinary singular tangles and three-page singular tangles are introduced in Sec. 3.1 and Sec. 3.2, respectively. The latter notion generalizes three-page embeddings of singular knots and helps us to prove Theorem 2. In Sec. 3.3, three-page tangles are classified (Theorem 4), and Theorem 2 then follows from Theorem 4 as a particular case. Theorem 4 is applied to prove Theorem 3 in Sec. 3.4. In Sec. 3.5, classification Theorem 5 for knuckle 4-valent graphs is stated. In Sec. 4 we deduce Lemma 3 which is used in the proof of Theorem 4.

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2. Three-Page Embeddings

2.1. Formal definition of three-page embeddings. An *arc* of a singular knot $K \subset \mathbb{R}^3$ at a point $A \in K$ is an arbitrary sufficiently small segment $J \subset K$ with endpoint A. Thus, exactly 4 arcs issue from each singular point. Let P_0, P_1, and P_2 be three half-planes in $\mathbb{R}^3$ with a common oriented boundary, $\partial P_0 = \partial P_1 = \partial P_2 = \alpha$. (All constructions in Secs. 2.1–2.2 are demonstrated in Fig. 2.) We set $\mathbb{Y} = P_0 \cup P_1 \cup P_2$ and call this union a *book with three pages*. An embedding of a singular knot K in the book $\mathbb{Y}$ is called a *three-page embedding* if the following conditions hold:

1) all singular points of K lie on the axis α;

2) *finiteness*: the intersection $K \cap \alpha = A_1 \cup \cdots \cup A_m$ is a finite nonempty point set;

3) at every nonsingular point $A_j \in K \cap \alpha$, two arcs lie in different half-planes;

4) a neighborhood of each singular point A_j lies in the plane $P_{i-1} \cup P_{i+1}$ for some $i \in \mathbb{Z}_3$;

5) *monotonicity*: for each $i \in \mathbb{Z}_3$, the restriction of the orthogonal projection $\mathbb{R}^3 \to \alpha \approx \mathbb{R}$ to each connected component of $K \cap P_i$ is a monotone function.

2.2. Construction of a three-page embedding from a plane diagram. Let D be a plane diagram of a singular knot K, i.e., a planar 4-valent graph with two types of vertices: the first type corresponds to singular points of K and the other corresponds to the usual *crossings* in a planar representation of K. Given a singular point B, let us mark two small arcs with endpoint B, namely a *singular bridge* L_B, that lie on different branches of the singular knot. Also, given a crossing of the diagram D, we mark a small segment (a *regular bridge*) of the overcrossing arc.

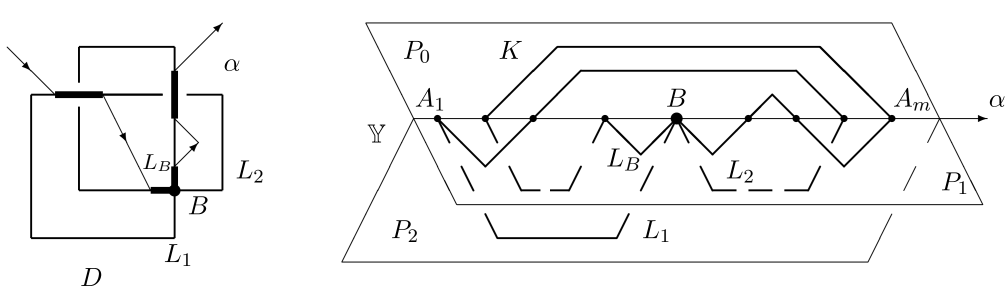

Fig. 2. Three-page embedding $K \subset \mathbb{Y}$, $w_K = a_0 a_1 b_2 b_0 x_0 b_2 d_2 c_1 c_2$

We then take a non-self-intersecting oriented path α in the plane of the diagram D with the following properties:

(1) the endpoints of the path α lie far from D;

(2) the path α traverses each bridge only once;

(3) *transversality*: the path α intersects the diagram of D transversally outside the bridges;

(4) *balance*: for an arbitrary singular point B, consider two unmarked arcs L_1 and L_2 with endpoint B that do not contain the singular bridge L_B; then the second endpoint of one of these arcs falls on the path α to the left of L_B and that of the other arc to the right of L_B.

Such a path α can be easily found as follows: consider only the bridges in the plane, i.e., finitely many arcs, and draw an arbitrary path α satisfying (1) and (2) through the bridges. Then the transversality property (3) will hold if the path α is made to be in general position with respect to the diagram D. Suppose that, for the resulting path, the balance property (4) does not hold for a singular point B, i.e., for example, both unmarked arcs L_1 and L_2 with endpoint B meet the path α to the left of the bridge L_B. Then we slightly perturb the path α to the right of L_B by a move similar to $R2$ in Fig. 1 in such a way that one of the two unmarked arcs L_1 or L_2 (this is the arc L_2 in Fig. 2) meets the path α to the right of L_B.

We now deform the plane of D in such a way that α becomes a straight line and the following *monotonicity* condition holds: the restriction of the orthogonal projection $\mathbb{R}^2 \to \alpha \approx \mathbb{R}$ to each connected component of $D - \alpha$ is a monotone function. We denote the upper half-plane above α by P_0 and the lower half-plane by P_2. Finally, we attach a third half-plane P_1 to α (on the reader's side) and push out all the bridges into P_1 according to the following rules. Each regular bridge becomes a trivial arc. Each singular bridge L becomes a "W"-like broken line such that the axis α meets it at its 3 upper vertices among which the middle one is the singular point B. In fact, a neighborhood of a singular point can be embedded in the plane $P_0 \cup P_2$ without pushing out the marked arcs into the third half-plane P_1. We have used the notion of a singular bridge to simplify the arguments.

2.3. Encoding three-page embeddings. Each three-page embedding of a singular knot is uniquely determined by its image in a small neighborhood of the axis α in the book $\mathbb{Y}$. Indeed, to reconstruct the whole embedding, it suffices to join the oppositely directed arcs in each half-plane beginning with the interior arcs. We always assume that the half-plane P_0 lies above the axis α and that the half-planes P_1 and P_2 are below α. Moreover, we suppose that P_1 is above P_2, i.e., the arcs in P_2 are shown in dashed lines. Only the 15 patterns in Fig. 3 may occur in a three-page embedding of a singular knot near the axis α.

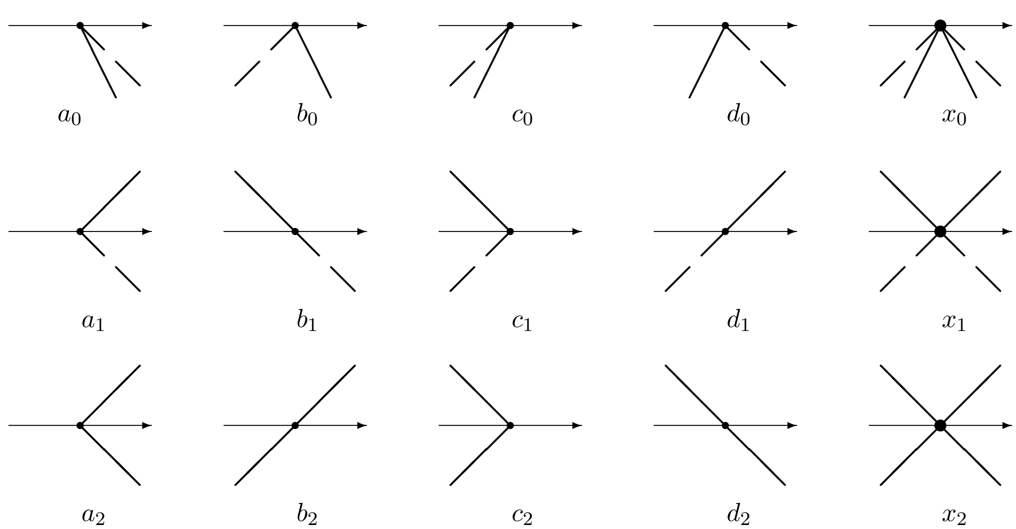

Fig. 3. Geometric interpretation of letters in the alphabet $\mathbb{A}$

Let W be the set of all words in the alphabet $\mathbb{A} = \{a_i, b_i, c_i, d_i, x_i,\ i \in \mathbb{Z}_3\}$ including the empty word $\varnothing$. For a given three-page embedding of the knot K, we consecutively write the letters of $\mathbb{A}$ corresponding to the intersection points in $K \cap \alpha$. We obtain some word $w_K \in W$ (Fig. 2).

2.4. Balanced words. Note that one cannot obtain all the words in W using the encoding in Sec. 2.3. A word is said to be *balanced* if it encodes some three-page embedding. The following simple geometric criterion for a word to be balanced exists: in each half-plane P_i all the arcs must be joined to one another. Arcs in an unbalanced three-page embedding can recede to infinity without meeting one another. One can easily restate this criterion algebraically in terms of the alphabet $\mathbb{A}$. For $i \in \mathbb{Z}_3$, a word w is said to be i-*balanced* if the substitution

$$a_i, b_i, c_i, d_i, x_i \to \varnothing, \quad a_{i\pm1}, b_{i-1}, d_{i+1} \to (, \quad b_{i+1}, c_{i\pm1}, d_{i-1} \to), \quad x_{i\pm1} \to)($$

results in an expression with *completely balanced brackets* (or, in another terminology, with *correctly placed brackets*). This means that, before each symbol, the number of left brackets is no less than that of the right ones, and their total numbers are equal. We denote by W_i the set of all i-balanced words in the alphabet $\mathbb{A}$. Then a word w is said to be *balanced* if it is i-balanced for each $i \in \mathbb{Z}_3$. Thus, the set of all balanced words is $W_b = W_0 \cap W_1 \cap W_2 \subset W$.

2.5. Proof of Theorem 1. We take a plane diagram D of a given singular knot K. Beginning with the diagram D, we construct a three-page embedding $K \subset \mathbb{Y}$ described in Sec. 2.2. Encode the resulting three-page embedding of K by a balanced word $w_K \in W_b$ according to the rules in Sec. 2.3. Finally, consider the word w_K as an element of the semigroup SK. □

3. Singular Tangles

3.1. Semigroup *ST* of singular tangles. To prove Theorem 2, we need the notion of a singular tangle. The category of tangles (without singularities) was studied by Turaev [21]. Take two horizontal half-lines $\mathbb{R}_+ \subset \mathbb{R}^3$ given, for example, by the coordinates, $(r, 0, 0)$ and $(r, 0, 1)$, where $r \in \mathbb{R}_+$. Mark the integer points $(j, 0, 0)$ and $(j, 0, 1)$ for all $j \in \mathbb{N}$ on both the half-lines. Let Γ be an arbitrary disconnected nonoriented infinite graph Γ with vertices of valency 1 and 4. A *singular tangle* is an embedding of Γ in the 3-dimensional layer $\{0 \leqslant z \leqslant 1\}$ such that (Fig. 4).

(1) the set of the 1-valent vertices of the graph Γ coincides with the set of marked points $\{(j, 0, 0), (j, 0, 1)\}_{j \in \mathbb{N}}$;

(2) all connected components of the graph Γ lying sufficiently far from the origin are the line segments joining the points $(k, 0, 0)$ and $(j, 0, 1)$ such that the difference $k - j$ is constant for all large j;

(3) there exists a plane neighborhood of each 4-valent vertex of the graph Γ.

We consider singular tangles up to ambient isotopy in the layer $\{0 \leqslant z \leqslant 1\}$ fixed on its boundary and such that condition (3) holds. Singular tangles can be represented by their plane diagrams by analogy with singular knots (Fig. 4). One can construct a *product* of singular tangles $\Gamma_1\Gamma_2$ by attaching the top half-line of Γ_2 to the bottom half-line of Γ_1. Thus, the isotopy classes of singular tangles form a semigroup ST. The *unit* in ST is the singular tangle consisting of vertical line segments. Let us introduce the singular tangles ξ_k, η_k, σ_k, σ_k^{-1}, τ_k $(k \in \mathbb{N})$ shown below.

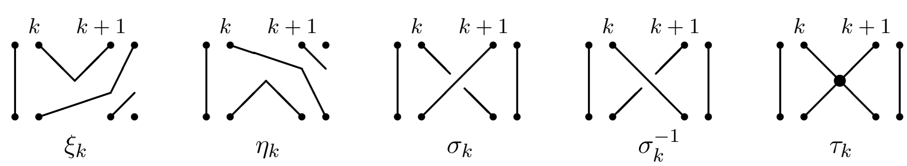

Fig. 4. Generators of the singular tangles

The following lemma transfers results of [21] from the classical case to ours.

Lemma 1. *The semigroup ST of singular tangles is generated by the elements ξ_k, η_k, σ_k, σ_k^{-1}, τ_k, $k \in \mathbb{N}$ (Fig. 4) and the following relations, where $k, l \in \mathbb{N}$:*

$$\xi_k\xi_l = \xi_{l+2}\xi_k, \quad \xi_k\eta_l = \eta_{l+2}\xi_k, \quad \xi_k\sigma_l = \sigma_{l+2}\xi_k, \quad \xi_k\tau_l = \tau_{l+2}\xi_k \qquad (l \geqslant k), \tag{11}$$

$$\eta_k\xi_l = \xi_{l-2}\eta_k, \quad \eta_k\eta_l = \eta_{l-2}\eta_k, \quad \eta_k\sigma_l = \sigma_{l-2}\eta_k, \quad \eta_k\tau_l = \tau_{l-2}\eta_k \qquad (l \geqslant k+2), \tag{12}$$

$$\sigma_k\xi_l = \xi_l\sigma_k, \quad \sigma_k\eta_l = \eta_l\sigma_k, \quad \sigma_k\sigma_l = \sigma_l\sigma_k, \quad \sigma_k\tau_l = \tau_l\sigma_k \qquad (l \geqslant k+2), \tag{13}$$

$$\tau_k\xi_l = \xi_l\tau_k, \quad \tau_k\eta_l = \eta_l\tau_k, \quad \tau_k\sigma_l = \sigma_l\tau_k, \quad \tau_k\tau_l = \tau_l\tau_k \qquad (l \geqslant k+2), \tag{14}$$

$$\eta_{k+1}\xi_k = 1 = \eta_k\xi_{k+1}, \tag{15}$$

$$\eta_{k+2}\sigma_{k+1}\xi_k = \sigma_k^{-1} = \eta_k\sigma_{k+1}\xi_{k+2}, \tag{16}$$

$$\eta_{k+2}\tau_{k+1}\xi_k = \tau_k = \eta_k\tau_{k+1}\xi_{k+2}, \tag{17}$$

$$\eta_k\sigma_k = \eta_k, \ \sigma_k\xi_k = \xi_k, \tag{18}$$

$$\sigma_k\sigma_k^{-1} = 1 = \sigma_k^{-1}\sigma_k, \tag{19}$$

$$\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}, \tag{20}$$

$$\sigma_k\sigma_{k+1}\tau_k = \tau_{k+1}\sigma_k\sigma_{k+1}, \tag{21}$$

$$\tau_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\tau_{k+1}, \tag{22}$$

$$\sigma_k\tau_k = \tau_k\sigma_k. \tag{23}$$

Proof. Recall that we work in the PL-category. This means that a given singular tangle in the layer $\{0 \leqslant z \leqslant 1\}$ consists of finite broken lines. The local maxima and minima of the height function on the components of a singular tangle are called *extremal points*. By *peculiarity* of a diagram of a tangle we mean either a 4-valent vertex or a crossing or an extremal point. We say that a singular tangle is in *general position* if its plane diagram satisfies the following conditions:

(1) the set of all peculiarities is finite;
(2) the crossings do not coincide with extremal points;
(3) for each 4-valent vertex, two arcs go up and the other two go down;
(4) each horizontal line (parallel to the Ox-axis) contains at most one peculiarity.

Obviously, every tangle can be moved to general position by a slight deformation. Then the tangle diagram is split by horizontal lines into strips each of which contains only one peculiarity. Considering the peculiarities in succession from top to bottom one by one, we write out the corresponding generators in Fig. 4 from left to right. Namely, the generators ξ_k and η_k represent extremal points, the generators σ_k and σ_k^{-1} correspond to crossings, and τ_k represents a 4-valent vertex. It remains to show that every ambient isotopy of singular tangles decomposes into "elementary isotopies" corresponding to relations (11)–(23). It follows from the Reidemeister theorem [15] and from considerations relating to the general position that an arbitrary isotopy of singular tangles in general position can be realized using the following moves:

(1) an isotopy in the class of diagrams in general position;
(2) an isotopy interchanging the vertical positions of two peculiarities;
(3) creation or annihilation of a pair of neighboring extremal points;
(4) an isotopy of a crossing or of a 4-valent vertex near an extremal point;
(5) the Reidemeister moves $R1$–$R5$ (Fig. 1).

A type (1) isotopy preserves the constructed word in the letters ξ_k,η_k, σ_k, σ_k^{-1}, τ_k, $k \in \mathbb{N}$. The type (2) isotopies are described by relations (11)–(14). The type (3) isotopies correspond to relations (15). In [21, proof of Lemma 3.4] it was shown that, in the smooth category, all isotopies of a crossing near an extremal point are purely geometrically decomposed into relations of the form (16). Similarly, in the PL-case under consideration, we can show that relations (17) are sufficient for the realization of the isotopies of a 4-valent vertex near an extremal point. Finally, Reidemeister moves $R1$–$R5$ correspond to relations (18)–(23), respectively. □

3.2. Three-page singular tangles. The notion of a three-page singular tangle will be used to prove Theorems 2 and 3. Consider three half-lines in the horizontal plane $\{z = 0\}$ that have a common endpoint. For example, let

$$T = \{x \geqslant 0, \, y = z = 0\} \cup \{y \geqslant 0, \, x = z = 0\} \cup \{x \leqslant 0, \, y = z = 0\} \subset \{z = 0\}.$$

We mark the integer points on the half-lines, $\{(j,0,0),(0,k,0),(-l,0,0) \mid j,k,l \in \mathbb{N}\}$. Let I be the line segment joining the points $(0,0,0)$ and $(0,0,1)$. We set

$$P_0 = \{x \geqslant 0,\ y = z = 0\} \times I, \quad P_1 = \{y \geqslant 0,\ x = z = 0\} \times I,$$
$$P_2 = \{x \leqslant 0,\ y = z = 0\} \times I.$$

Here P_i (where $i \in \mathbb{Z}_3$) is not formally half-plane, but a strip $I \times \mathbb{R}$, which we will call a *page*. The book $\mathbb{Y}$ in Sec. 2.1 is the interior of the set $T \times I$, i.e., in Sec. 2 we considered the embeddings $K \subset T \times I$ such that the knot K does not intersect the boundary half-lines. Let Γ be an arbitrary disconnected nonoriented infinite graph with vertices of valency 1, 2, and 4. A *three-page singular tangle* is an embedding of Γ in $T \times I$ such that (Fig. 5)

(1) the set of 1-valent vertices of the graph Γ coincides with the set of the marked points

$$\{(j,0,0),(j,0,1),(0,k,0),(0,k,1),(-l,0,0),(-l,0,1) \mid j,k,l \in \mathbb{N}\};$$

(2) all 4-valent vertices of Γ lie in the segment I;

(3) *finiteness*: the intersection $\Gamma \cap I = A_1 \cup \dots \cup A_m$ is a finite point set;

(4) the two arcs at each 2-valent vertex $A_j \in \Gamma \cap I$ lie in different half-planes;

(5) a neighborhood of each 4-valent vertex of Γ lies in exactly two pages among P_0, P_1, and P_2;

(6) *monotonicity*: for every $i \in \mathbb{Z}_3$, the restriction of the orthogonal projection $T \times I \to I \approx [0,1]$ to each connected component of $\Gamma \cap P_i$ is a monotone function;

(7) for each $i \in \mathbb{Z}_3$, all connected components of the graph Γ lying in the plane P_i sufficiently far from the origin are parallel line segments.

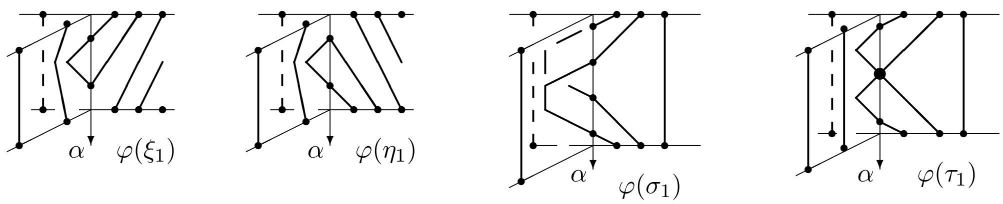

Fig. 5. Three-page singular tangles

As in the case of singular tangles in Sec. 3.1, isotopy classes of three-page tangles in the layer $\{0 \leqslant z \leqslant 1\}$ form a semigroup. Each three-page tangle can be encoded by a word in the alphabet $\mathbb{A} = \{a_i, b_i, c_i, d_i, x_i,\ i \in \mathbb{Z}_3\}$ (Fig. 3) in exactly the same way as in Sec. 2.3. We now define a subsemigroup of three-page tangles that is isomorphic to the semigroup ST introduced in Sec. 3.1. A three-page tangle is said to be *almost balanced* if the corresponding word in the alphabet $\mathbb{A}$ is 1-balanced and 2-balanced (see Sec. 2.4). Note that, for any i-balanced three-page tangle, all line segments joining the marked points in the page P_i can be assumed to be vertical. We denote by BT the semigroup of almost balanced three-page tangles. We define a map $\varphi : ST \to BT$ on the generators as follows (Fig. 5):

$$\varphi(\xi_k) = d_2^k c_2 b_2^{k-1}, \quad \varphi(\eta_k) = d_2^{k-1} a_2 b_2^k, \quad \varphi(\sigma_k) = d_2^{k-1} b_1 d_2 d_1 b_2^k,$$
$$\varphi(\sigma_k^{-1}) = d_2^k b_1 b_2 d_1 b_2^{k-1}, \quad \varphi(\tau_k) = d_2^k x_2 b_2^k, \qquad k \in \mathbb{N}. \tag{24}$$

Each tangle goes to the corresponding three-page embedding plus vertical line segments. The following Lemma is proved in the same way as [11, Lemma 3].

Lemma 2. *The map $\varphi \colon ST \to BT$ is a well-defined isomorphism of semigroups.*

Proof. First, let us show that isotopy equivalent singular tangles go into isotopy equivalent three-page singular tangles under the map φ. Actually, by definition, singular tangles in the semigroups ST and BT are considered up to isotopy in the layer $\{0 < z < 1\}$. It follows that the map φ is injective. We now construct the inverse map $\psi \colon BT \to ST$. Let us associate with each almost balanced three-page tangle $\Gamma \in BT$ a singular tangle $\psi(\Gamma) \in ST$ given by the following diagram. According to the property of being almost balanced, we assume that all line segments of Γ lying in the pages P_1 and P_2 are vertical. Deleting all these vertical line segments from Γ, we obtain a singular tangle $\psi(\Gamma)$ in the sense of Sec. 3.1. Clearly, the composition $\psi \circ \varphi \colon ST \to ST$ is identical on the generators (Fig. 5). Therefore the maps φ and ψ are mutually inverse. □

3.3. Classification of three-page singular tangles. We denote by $\varphi(11)$–$\varphi(23)$ the relations between words in the alphabet $\mathbb{A}$ which are obtained from relations (11)–(23) of the semigroup ST under the isomorphism $\varphi \colon ST \to BT$ (Lemma 2). The following lemma will be proved in Sec. 4.3.

Lemma 3. *Relations $\varphi(11)$–$\varphi(23)$ follow from relations (1)–(10) of the semigroup SK.*

Theorem 2 is a special case of the following classification theorem for three-page singular tangles, which we prove by analogy with Theorem 1 of [11].

Theorem 4. *The semigroup of the isotopy classes of three-page singular tangles is isomorphic to the semigroup SK.*

Proof. As was already mentioned in Sec. 3.2, with each three-page singular tangle, a word in the alphabet $\mathbb{A}$ and hence an element of the semigroup SK can be associated. Conversely, each element of the semigroup SK can be completed to form a three-page tangle by adding three families of parallel line segments on each page P_i, $i \in \mathbb{Z}_3$. For example, the three-page tangles represented in Fig. 3 correspond to the following elements in SK: d_2c_2, a_2b_2, $b_1d_2d_1b_2$, $d_2x_2b_2$. Relations (1)–(10) of the semigroup SK can be easily realized by ambient isotopies in the layer $\{0 < z < 1\}$.

Therefore it remains to prove that each isotopy of three-page singular tangle can be decomposed into "elementary isotopies" corresponding to relations (1)–(10) of SK. It suffices to do this for almost balanced three-page tangles. Indeed, for a given three-page tangle corresponding to a word $w \in W$, let n_1 and m_1 (n_2 and m_2) be the maximal indices of points on the half-lines $P_1 \cap \{z = 0\}$ and $P_1 \cap \{z = 1\}$ ($P_2 \cap \{z = 0\}$ and $P_2 \cap \{z = 1\}$) that are joined by arcs with points on the line segment I. For example, for the tangle $w = a_0$, these are $n_1 = n_2 = 0$ and $m_1 = m_2 = 1$. For all almost balanced tangles (for example, see Fig. 5), we have $n_1 = m_1 = n_2 = m_2 = 0$. Then the word $b_1^{n_2} d_2^{n_1} w b_2^{m_1} d_1^{m_2}$ is almost balanced. Because of the invertibility of the generators b_i and d_i, such a transformation and its inverse send equivalent words to equivalent ones.

By Lemma 2, we can associate a singular tangle in the sense of Sec. 3.1 with each almost balanced word. For such tangles, each isotopy is already decomposed into the elementary isotopies corresponding to relations $\varphi(11)$–$\varphi(23)$ (see Lemmas 1 and 2). Thus, Lemma 3 completes the proof of Theorem 4. □

3.4. Proof of Theorem 3. We will identify an arbitrary three-page singular tangle with the corresponding element of the semigroup SK. A three-page tangle is said to be *knot-like* if it contains a singular knot near the axis α, and the rest of it consists only of vertical segments. Clearly, the knot-like tangles correspond to the balanced words in $W_b \subset W$.

Lemma 4. *An element $w \in SK$ defines a knot-like tangle if and only if w is a central element in the semigroup SK.*

Proof. The "only if" part is geometrically evident, namely, a singular knot can be moved by an isotopy to any place in a given tangle, i.e., by Theorem 4, a knot-like element commutes with any other element. Let w be a central element in SK. Then, for each $k \in \mathbb{N}$, we have $b_i^k d_i^k w = w b_i^k d_i^k$. Denote by m (n) the number of arcs of the three-page tangle w that recede, in the page P_{i-1}, to the left (to the right) boundary. Then, for a sufficiently large k, the number of the arcs of the three-page tangle $b_i^k d_i^k w$ that recede, in the page P_{i-1}, to the left boundary is equal to k, and for the tangle $w b_i^k d_i^k$, it is equal to $m + k - n$, i.e., $m = n$. Hence, for a sufficiently large l and an arbitrary $j = i - 1 \in \mathbb{Z}_3$, the word $a_0^l a_1^l w c_1^l c_0^l$ is j-balanced, i.e., it is balanced. Since w is a central element, the word $w a_0^l a_1^l c_1^l c_0^l$ is also balanced. Consequently, it is geometrically obvious that the element w determines a knot-like tangle. □

Theorem 3 follows from Lemma 4.

3.5. Classification of knuckle 4-valent graphs. The isotopy classification problem of such graphs was considered in [16]. Dynnikov's method makes it possible to solve the problem by analogy with the case of singular knots. Let us introduce a semigroup FG having the same generators and relations as the semigroup SK with the only distinction that relation (6) is replaced by

$$x_i(d_{i+1}d_id_{i-1}) = x_i. \tag{6'}$$

Then the semigroup FG has 15 generators and 84 defining relations.

Theorem 5. *The center of the semigroup FG classifies all nonoriented knuckle 4-valent graphs in $\mathbb{R}^3$ up to ambient isotopy.*

Theorem 5 is proved analogously to Theorems 1–3 with replacement of relation (6) by (6′) and relation (23) in Lemma 1 by

$$\sigma_k\tau_k = \tau_k. \tag{23'}$$

4. Proof of Lemma 3

Bellow in Proposition 1 we derive new word equivalencies from relations (1)–(10) of the semigroup SK. In Sec. 4.2, using Propositions 1–3, we prove Lemma 5 on the decomposition of an arbitrary i-balanced word. Lemma 5 and Proposition 6 reduce the infinite set of relations $\varphi(11)$–$\varphi(23)$ to finitely many relations (1)–(10). The proof of Lemma 3 is completed in Sec. 4.3 using Propositions 5 and 6. All relations will be obtained in a formal way, but they have a geometric interpretation (Fig. 3).

4.1. Corollaries of relations (1)–(10).

Proposition 1. *Equivalencies* (1)–(10) *imply the following word equivalences* (*it is everywhere assumed that $i \in \mathbb{Z}_3$ and $w_i \in \mathbb{B}_i = \{a_i, b_i, c_i, d_i, b_{i-1}b_id_{i-1}, b_{i-1}d_id_{i-1}\}$*):

$$\begin{gathered} b_i \sim d_{i+1}d_{i-1}, \quad \textit{or} \quad b_{i+1} \sim d_{i-1}d_i, \quad b_{i-1} \sim d_id_{i+1}, \\ \textit{or} \quad b_0 \sim d_1d_2, \quad b_1 \sim d_2d_0, \quad b_2 \sim d_0d_1, \end{gathered} \tag{25}$$

$$\begin{gathered} d_i \sim b_{i-1}b_{i+1}, \quad \textit{or} \quad d_{i-1} \sim b_{i+1}b_i, \quad d_{i+1} \sim b_ib_{i-1}, \\ \textit{or} \quad d_0 \sim b_2b_1, \quad d_1 \sim b_0b_2, \quad d_2 \sim b_1b_0, \end{gathered} \tag{26}$$

$$d_{i+1}b_{i-1} \sim b_{i-1}d_{i+1}t_i, \quad b_{i+1}d_{i-1} \sim t_id_{i-1}b_{i+1}, \quad \textit{where } t_i = b_{i+1}d_{i-1}d_{i+1}b_{i-1}, \tag{27}$$

$$a_i \sim a_{i-1}b_{i+1}, \quad c_i \sim d_{i+1}c_{i-1}, \tag{28}$$

$$a_ib_i \sim a_{i-1}d_{i-1}, \quad d_ic_i \sim b_{i-1}c_{i-1}, \tag{29}$$

$$b_i \sim a_ib_ic_i, \quad d_i \sim a_id_ic_i, \tag{30}$$

$$b_{i-1}x_{i+1}d_{i-1} \sim x_i, \tag{31}$$

$$b_ix_id_i \sim d_{i+1}x_{i+1}b_{i+1}, \tag{32}$$

$$(d_ic_i)w_{i+1} \sim w_{i+1}(d_ic_i), \tag{33}$$

$$(b_ic_i)w_{i-1} \sim w_{i-1}(b_ic_i), \tag{34}$$

$$(a_ib_i)w_{i+1} \sim w_{i+1}(a_ib_i), \tag{35}$$

$$(a_id_i)w_{i-1} \sim w_{i-1}(a_id_i), \tag{36}$$

$$t_iw_i \sim w_it_i, \quad t'_iw_i \sim w_it'_i, \quad \textit{where } t_i = b_{i+1}d_{i-1}d_{i+1}b_{i-1},\ t'_i = d_{i-1}b_{i+1}b_{i-1}d_{i+1}, \tag{37}$$

$$(d_ix_ib_i)w_{i+1} \sim w_{i+1}(d_ix_ib_i), \tag{38}$$

$$(b_ix_id_i)w_{i-1} \sim w_{i-1}(b_ix_id_i), \tag{39}$$

$$d_{i+1}b_{i-1}w_id_{i-1}b_{i+1} \sim b_{i-1}d_{i+1}w_ib_{i+1}d_{i-1}, \tag{40}$$

$$b_{i-1}^2a_id_{i-1}^2 \sim (b_{i-1}a_id_{i-1})d_i^2(b_{i-1}b_id_{i-1})b_i, \tag{41}$$

$$b_{i-1}^2c_id_{i-1}^2 \sim d_i(b_{i-1}d_id_{i-1})b_i^2(b_{i-1}c_id_{i-1}), \tag{42}$$

$$b_{i-1}^2b_id_{i-1}^2 \sim (b_{i-1}b_id_{i-1})d_i^2(b_{i-1}b_id_{i-1})b_i, \tag{43}$$

$$b_{i-1}^2d_id_{i-1}^2 \sim d_i(b_{i-1}d_id_{i-1})b_i^2(b_{i-1}d_id_{i-1}), \tag{44}$$

$$b_{i-1}^2x_id_{i-1}^2 \sim (b_{i-1}^2b_id_{i-1}^2)(b_{i-1}d_id_{i-1})d_i^2x_ib_i^2(b_{i-1}b_id_{i-1})(b_{i-1}^2d_id_{i-1}^2). \tag{45}$$

Proof. Note that equivalencies (25)–(27) readily follow from (3)–(4). By (4), we have $d_i \sim b_i^{-1}$, $b_{i-1}b_id_{i-1} \sim (b_{i-1}d_id_{i-1})^{-1}$ and $t_i' \sim t_i^{-1}$. Therefore (35), (37), and (38) immediately follow from (8), (9), and (10), respectively. The other equivalencies will be consecutively established using those already proved. Recall that $i \in \mathbb{Z}_3 = \{0,1,2\}$, i.e., we have $(i+1)+1 = i-1$ and $(i-1)-1 = i+1$ in $\mathbb{Z}_3$.

(28): $a_{i-1}b_{i+1} \overset{(1)}{\sim} (a_id_{i+1})b_{i+1} \overset{(4)}{\sim} a_i, \quad d_{i+1}c_{i-1} \overset{(1)}{\sim} d_{i+1}(b_{i+1}c_i) \overset{(4)}{\sim} c_i,$

(29): $a_ib_i \overset{(28)}{\sim} (a_{i-1}b_{i+1})b_i \overset{(26)}{\sim} a_{i-1}d_{i-1}, \quad d_ic_i \overset{(26)}{\sim} (b_{i-1}b_{i+1})c_i \overset{(1)}{\sim} b_{i-1}c_{i-1},$

(30): $a_ib_ic_i \overset{(29)}{\sim} a_i(d_{i+1}c_{i+1}) \overset{(1)}{\sim} a_{i-1}c_{i+1} \overset{(1)}{\sim} b_i, \quad a_id_ic_i \overset{(29)}{\sim} a_i(b_{i-1}c_{i-1}) \overset{(28)}{\sim} a_{i+1}c_{i-1} \overset{(1)}{\sim} d_i,$

(31): $b_{i-1}x_{i+1}d_{i-1} \overset{(2)}{\sim} b_{i-1}(d_{i-1}x_ib_{i-1})d_{i-1} \overset{(4)}{\sim} x_i,$

(32): $b_ix_id_i \overset{(31)}{\sim} b_i(b_{i-1}x_{i+1}d_{i-1})d_i \overset{(25),(26)}{\sim} d_{i+1}x_{i+1}b_{i+1}.$

Below, in the proof of (33), we first commute b_{i+1} and d_ic_i and then we use this relation to commute a_{i+1} and d_ic_i.

(33*b*): $b_{i+1}(d_ic_i) \overset{(30)}{\sim} (a_{i+1}b_{i+1}c_{i+1})(d_ic_i) \overset{(7)}{\sim} a_{i+1}b_{i+1}(d_ic_i)c_{i+1} \overset{(26)}{\sim} a_{i+1}b_{i+1}(b_{i-1}b_{i+1})c_ic_{i+1}$
$\overset{(1)}{\sim} (a_{i+1}b_{i+1})b_{i-1}c_{i-1}c_{i+1} \overset{(8)}{\sim} b_{i-1}c_{i-1}(a_{i+1}b_{i+1})c_{i+1} \overset{(30)}{\sim} b_{i-1}c_{i-1}b_{i+1} \overset{(1)}{\sim} b_{i-1}(b_{i+1}c_i)b_{i+1}$
$\overset{(26)}{\sim} (d_ic_i)b_{i+1},$

(33*a*): $a_{i+1}(d_ic_i) \overset{(1)}{\sim} (a_{i-1}d_i)(d_ic_i) \overset{(26)}{\sim} a_{i-1}(b_{i-1}b_{i+1})(d_ic_i) \overset{(33b)}{\sim} a_{i-1}b_{i-1}(d_ic_i)b_{i+1}$
$\overset{(35)}{\sim} (d_ic_i)(a_{i-1}b_{i-1})b_{i+1} \overset{(26)}{\sim} (d_ic_i)(a_{i-1}d_i) \overset{(1)}{\sim} (d_ic_i)a_{i+1}.$

The other equivalences in (33) follow from (33a), (33b) and (7). Equivalences (34), (36),(39) follow from (29) and (33), (29) and (35), and (32) and (38), respectively. The remaining calculations are trivial,

(40): $d_{i+1}b_{i-1}w_id_{i-1}b_{i+1} \overset{(27)}{\sim} (b_{i-1}d_{i+1}t_i)w_id_{i-1}b_{i+1} \overset{(37)}{\sim} b_{i-1}d_{i+1}(w_it_i)d_{i-1}b_{i+1} \overset{(27)}{\sim} b_{i-1}d_{i+1}w_ib_{i+1}d_{i-1},$

(41): $b_{i-1}^2a_id_{i-1}^2 \overset{(4)}{\sim} b_{i-1}^2a_i(d_ib_i)d_{i-1}^2 \overset{(36)}{\sim} b_{i-1}(a_id_i)(b_{i-1}b_i)d_{i-1}^2 \overset{(26)}{\sim} b_{i-1}a_id_i(b_{i-1}b_i)d_{i-1}(b_{i+1}b_i)$
$\overset{(37)}{\sim} b_{i-1}a_ib_{i+1}(d_ib_{i-1}b_id_{i-1})b_i \overset{(25)}{\sim} (b_{i-1}a_id_{i-1})d_i^2(b_{i-1}b_id_{i-1})b_i,$

(42): $b_{i-1}^2b_id_{i-1}^2 \overset{(4)}{\sim} b_{i-1}(b_id_i)b_{i-1}b_id_{i-1}^2 \overset{(26)}{\sim} b_{i-1}b_i(d_ib_{i-1}b_id_{i-1})(b_{i+1}b_i) \overset{(37)}{\sim} b_{i-1}b_ib_{i+1}(d_ib_{i-1}b_id_{i-1})b_i$
$\overset{(25)}{\sim} (b_{i-1}b_id_{i-1})d_i^2(b_{i-1}b_id_{i-1})b_i,$

(43): $b_{i-1}^2c_id_{i-1}^2 \overset{(4)}{\sim} b_{i-1}^2(d_ib_i)c_id_{i-1}^2 \overset{(34)}{\sim} b_{i-1}^2d_id_{i-1}(b_ic_i)d_{i-1} \overset{(25)}{\sim} (d_id_{i+1})(b_{i-1}d_id_{i-1}b_i)c_id_{i-1}$
$\overset{(37)}{\sim} d_i(b_{i-1}d_id_{i-1}b_i)d_{i+1}c_id_{i-1} \overset{(26)}{\sim} d_i(b_{i-1}d_id_{i-1})b_i^2(b_{i-1}c_id_{i-1}),$

(44): $b_{i-1}^2d_id_{i-1}^2 \overset{(4)}{\sim} b_{i-1}^2d_id_{i-1}(b_id_i)d_{i-1} \overset{(25)}{\sim} (d_id_{i+1})b_{i-1}d_id_{i-1}b_id_id_{i-1} \overset{(37)}{\sim} d_i(b_{i-1}d_id_{i-1}b_i)d_{i+1}d_id_{i-1}$
$\overset{(26)}{\sim} d_i(b_{i-1}d_id_{i-1})b_i^2(b_{i-1}d_id_{i-1}),$

(45): $b_{i-1}^2x_id_{i-1}^2 \overset{(4)}{\sim} b_{i-1}^2(b_id_i)x_i(b_ib_{i+1}^2d_{i+1}^2d_i)d_{i-1}^2 \overset{(10)}{\sim} b_{i-1}^2b_ib_{i+1}^2(d_ix_ib_i)d_{i+1}^2d_id_{i-1}^2$
$\overset{(25),(26)}{\sim} b_{i-1}^2b_i(d_{i-1}d_i)^2d_ix_ib_i(b_ib_{i-1})^2d_id_{i-1}^2$
$\overset{(4)}{\sim} (b_{i-1}^2b_id_{i-1}^2)(b_{i-1}d_id_{i-1})d_i^2x_ib_i^2(b_{i-1}b_id_{i-1})(b_{i-1}^2d_id_{i-1}^2).$ □

4.2. Decomposition of *i*-balanced words.

Proposition 2. *For each $i \in \mathbb{Z}_3$, every i-balanced word is equivalent to an i-balanced word containing only the letters a_i, b_i, c_i, d_i, x_i, b_{i-1}, and d_{i-1}.*

Proof. Using the substitutions below, we can eliminate the other letters,

$$a_{i-1} \overset{(1)}{\sim} a_id_{i+1}, \quad a_{i+1} \overset{(28)}{\sim} a_ib_{i-1}, \quad c_{i-1} \overset{(1)}{\sim} b_{i+1}c_i, \quad c_{i+1} \overset{(28)}{\sim} d_{i-1}c_i,$$
$$b_{i+1} \overset{(25)}{\sim} d_{i-1}d_i, \quad d_{i+1} \overset{(26)}{\sim} b_ib_{i-1}, \quad x_{i-1} \overset{(2)}{\sim} d_ix_{i+1}b_i, \quad x_{i+1} \overset{(2)}{\sim} d_{i-1}x_ib_{i-1}. \quad \square$$

In what follows, fix an index $i \in \mathbb{Z}_3$. Let w be an i-balanced word on the letters a_i, b_i, c_i, d_i, x_i, b_{i-1}, and d_{i-1}. We consider the substitution $\mu\colon a_i, b_i, c_i, d_i, x_i \to \bullet$, $b_{i-1} \to ($, $d_{i-1} \to)$. Denote by $\mu(w)$ the resulting *encoding* consisting of brackets and bullets. Since the given word w is i-balanced, the encoding $\mu(w)$ (without bullets) is a balanced bracket expression. For each place k, denote by $\mathrm{dif}(k)$ the difference between the number of the left and right brackets in a subword of $\mu(w)$ ending at this place. The maximum of $\mathrm{dif}(k)$ over all k will be called the *depth* of w, $d(w)$. For example, the word $w = b_{i-1}^2 a_i d_{i-1}^2$ has the encoding $\mu(w) = ((\bullet))$ and the depth $d(w) = 2$.

By the *star of depth* k, we mean an encoding of the type $(^k\bullet)^k$ which has k pairs of brackets. The bullet is a star of depth 0. If, for a word w, its encoding $\mu(w)$ decomposes into several stars, then w is said to be *star decomposable*. In this case, the depth $d(w)$ is maximal among the depths of all stars participating in the decomposition.

Proposition 3. *Every i-balanced word w is equivalent to a star decomposable word w' of the same depth.*

Proof. Consider the beginning of the encoding $\mu(w)$. After several initial left brackets, $\mu(w)$ contains either a right bracket or a bullet. In the first case, we delete a pair of brackets $()$ by the rule $b_{i-1}d_{i-1} \overset{(4)}{\sim} \varnothing$. Hence, we can assume that the next symbol after the sequence of k left brackets is a bullet. Since $\mu(w)$ is balanced, after this bullet there can be a sequence of j, $0 \leqslant j \leqslant k$, right brackets. If $j < k$, then we insert the subword $d_{i-1}^{k-j} b_{i-1}^{k-j} \overset{(4)}{\sim} \varnothing$ in w after the last right bracket. This operation does not change the depth $d(w)$. Therefore, for the resulting word w_1, the encoding $\mu(w_1)$ contains a star of depth k at the beginning. Continuing this process, after a finite number of steps, we get a star decomposable word w_N of the same depth $d(w_N) = d(w)$. □

For an arbitrary letter s, we denote by s' the word $b_{i-1} s d_{i-1}$, for example, $a_i' = b_{i-1} a_i d_{i-1}$.

Proposition 4. *Each star decomposable word w is equivalent to a word decomposed into the following i-balanced subwords*: a_i, b_i, c_i, d_i, x_i, a_i', b_i', c_i', d_i', x_i'.

Proof. We use induction on the depth $d(w)$. The case $d(w) = 1$ is trivial. Let $\mu(w)$ contain a star of depth $k \geqslant 2$. Let us apply one of the following transformations to every such star:

$$
\begin{aligned}
&u = b_{i-1}^2 a_i d_{i-1}^2 \overset{(41)}{\sim} a_i' d_i^2 b_i' b_i = v, \quad \text{i.e., } \mu(u) = ((\bullet)) \to \mu(v) = (\bullet)\bullet\bullet(\bullet)\bullet,\\
&u = b_{i-1}^2 b_i d_{i-1}^2 \overset{(42)}{\sim} b_i' d_i^2 b_i' b_i = v, \quad \text{i.e., } \mu(u) = ((\bullet)) \to \mu(v) = (\bullet)\bullet\bullet(\bullet)\bullet,\\
&u = b_{i-1}^2 c_i d_{i-1}^2 \overset{(43)}{\sim} d_i d_i' b_i^2 c_i' = v, \quad \text{i.e., } \mu(u) = ((\bullet)) \to \mu(v) = \bullet(\bullet)\bullet\bullet(\bullet),\\
&u = b_{i-1}^2 d_i d_{i-1}^2 \overset{(44)}{\sim} d_i d_i' b_i^2 d_i' = v, \quad \text{i.e., } \mu(u) = ((\bullet)) \to \mu(v) = \bullet(\bullet)\bullet\bullet(\bullet),\\
&u = b_{i-1}^2 x_i d_{i-1}^2 \overset{(45)}{\sim} (b_{i-1}^2 b_i d_{i-1}^2)(b_{i-1} d_i d_{i-1}) d_i^2 x_i b_i^2 (b_{i-1} b_i d_{i-1})(b_{i-1}^2 d_i d_{i-1}^2)\\
&\qquad \overset{(43{,}44)}{\sim} (b_i' d_i^2 b_i' b_i) d_i' d_i^2 x_i b_i^2 b_i' (d_i d_i' b_i^2 d_i') = v,\\
&\qquad \text{i.e., } \mu(u) = ((\bullet)) \to \mu(v) = (\bullet)\bullet\bullet(\bullet)\bullet(\bullet)\bullet\bullet\bullet\bullet\bullet(\bullet)\bullet(\bullet)\bullet\bullet(\bullet).
\end{aligned}
$$

We get a word $w_1 \sim w$, which, according to Proposition 3, is equivalent to a star decomposed word w_2 of depth $d(w_2) = d(w) - 1$. The induction step is complete. □

Lemma 5. *For all $i \in \mathbb{Z}_3$, each i-balanced word is equivalent to a word which can be decomposed into the i-balanced words belonging to the set $\mathbb{B}_i = \{a_i, b_i, c_i, d_i, x_i, b_{i-1}b_id_{i-1}, b_{i-1}d_id_{i-1}\}$.*

Proof. By Propositions 3 and 4, it remains to eliminate only the following words:

$$
\begin{aligned}
a_i' &= b_{i-1} a_i d_{i-1} \overset{(25)}{\sim} (d_i d_{i+1}) a_i d_{i-1} \overset{(4)}{\sim} d_i d_{i+1} a_i (b_i d_i) d_{i-1} \overset{(35)}{\sim} d_i (a_i b_i) d_{i+1} d_i d_{i-1} \overset{(26)}{\sim} d_i a_i b_i^2 (b_{i-1} d_i d_{i-1}),\\
c_i' &= b_{i-1} c_i d_{i-1} \overset{(26)}{\sim} b_{i-1} c_i (b_{i+1} b_i) \overset{(4)}{\sim} b_{i-1} (b_i d_i c_i) b_{i+1} b_i \overset{(33)}{\sim} b_{i-1} b_i b_{i+1} (d_i c_i) b_i \overset{(25)}{\sim} (b_{i-1} b_i d_{i-1}) d_i^2 c_i b_i,
\end{aligned}
$$

$$x_i' = b_{i-1}x_id_{i-1} \overset{(4)}{\sim} b_{i-1}(b_ib_{i+1}d_{i+1}d_i)x_i(b_id_i)d_{i-1} \overset{(38)}{\sim} b_{i-1}b_ib_{i+1}(d_ix_ib_i)d_{i+1}d_id_{i-1}$$
$$\overset{(25)}{\sim} (b_{i-1}b_id_{i-1})d_i^2x_ib_id_{i+1}d_id_{i-1} \overset{(26)}{\sim} (b_{i-1}b_id_{i-1})d_i^2x_ib_i^2(b_{i-1}d_id_{i-1}). \qquad \square$$

Denote by (33′)–(40′) relations (33)–(40) assuming that $w_i \in W_i$.

Proposition 5. *Generalized equivalences* (33′)–(40′) *hold for arbitrary i-balanced words $w_i \in W_i$.*

Proof. By Lemma 5, every i-balanced word in W_i can be decomposed into the elementary words belonging to $\mathbb{B}_i$. Since equivalences (33)–(40) hold for the words in $\mathbb{B}_i$, they also hold for the words in W_i. Note that we get an infinitely many new equivalences (33′)–(40′). $\square$

4.3. Deduction of relations $\varphi(11)$–$\varphi(23)$ from relations (1)–(10) of the semigroup SK. For each $l \geqslant 1$, denote by u_l a symbol in the set of generators $\{\xi_l, \eta_l, \sigma_l, \sigma_l^{-1}, \tau_l\}$ of the semigroup ST of singular tangles. We define the *shift maps* $\theta_k\colon ST \to ST$ and $\rho_k\colon BT \to BT$ using the formulas $\theta_k(u_l) = u_{k+l}$ and $\rho_k(w) = d_2^kwb_2^k$. Evidently, the shift map $\theta_k\colon ST \to ST$ is a well-defined homomorphism. Indeed, each relation in (11)–(23) for $k > 1$ is obtained from the corresponding relation for $k = 1$ by the shift map θ_{k-1}. For example, relation $\xi_k\xi_l = \xi_{l+2}\xi_k$ is obtained from $\xi_1\xi_{l-k+1} = \xi_{l-k+3}\xi_1$ by the shift map θ_{k-1}. By (4) the shift map ρ_k sends equivalent words to equivalent ones, i.e., ρ_k is also a homomorphism. Moreover, the following diagram

$$\begin{array}{ccc} ST & \xrightarrow{\theta_k} & ST \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ BT & \xrightarrow{\rho_k} & BT \end{array}$$

is commutative, which implies the following proposition.

Proposition 6. *For each $k \in \mathbb{N}$, relations* $\varphi(11)$–$\varphi(23)$ *can be obtained from relations* $\varphi(11)$–$\varphi(23)$ *for $k = 1$ using the equivalences $b_2d_2 \sim 1 \sim d_2b_2$ of* (4). $\square$

Proof of Lemma 3. Here we deduce relations $\varphi(11)$–$\varphi(23)$ of the semigroup BT in the alphabet $\mathbb{A}$ from relations (1)–(10) and (25)–(40). We use generalized equivalences (33′)–(40′) in Proposition 5. We denote by the star $\star$ the following images under the map φ for $k = 1$:

$$\varphi(\xi_1) = d_2c_2, \quad \varphi(\eta_1) = a_2b_2, \quad \varphi(\sigma_1) = b_1d_2d_1b_2,$$
$$\varphi(\sigma_1^{-1}) = d_2b_1b_2d_1, \quad \varphi(\tau_1) = d_2x_2b_2. \tag{24′}$$

Then, by (24), we have $\varphi(u_l) = d_2^{l-1} \star b_2^{l-1}$. Note that the words $d_2^l \star b_2^l \in W$ are 1-balanced and 2-balanced (Fig. 5). Therefore relations $\varphi(11)$–$\varphi(14)$ can be proved following the same scheme,

(11): $\varphi(\xi_1u_l) \overset{(24)}{=} (d_2c_2)(d_2^{l-1} \star b_2^{l-1}) \overset{(4)}{\sim} d_2^2(b_2c_2)(d_2^{l-1} \star b_2^{l-1}) \overset{(34')}{\sim} d_2^2(d_2^{l-1} \star b_2^{l-1})(b_2c_2) \overset{(4)}{\sim} \varphi(u_{l+2}\xi_1)$,

(12): $\varphi(\eta_1u_l) \overset{(24)}{=} (a_2b_2)(d_2^{l-1} \star b_2^{l-1}) \overset{(4)}{\sim} (a_2d_2)(d_2^{l-3} \star b_2^{l-3})b_2^2 \overset{(36')}{\sim} (d_2^{l-3} \star b_2^{l-3})(a_2d_2)b_2^2 \overset{(4)}{\sim} \varphi(u_{l-2}\eta_1)$,

(13): $\varphi(\sigma_1u_l) \overset{(24)}{=} (b_1d_2d_1b_2)(d_2^{l-1} \star b_2^{l-1}) \overset{(4)}{\sim} (b_1d_2d_1)(d_2^{l-2} \star b_2^{l-1}) \overset{(25),(2)}{\sim} d_2^2(b_2d_0d_2b_0)(d_2^{l-3} \star b_2^{l-3})b_2^2$
$\overset{(37')}{\sim} d_2^2(d_2^{l-3} \star b_2^{l-3})(b_2d_0d_2b_0)b_2^2 \overset{(4),(26)}{\sim} (d_2^{l-1} \star b_2^{l-2})(b_2b_1)d_2d_1b_2 \overset{(24)}{=} \varphi(u_l\sigma_1)$,

(14): $\varphi(\tau_1u_l) \overset{(24)}{=} (d_2x_2b_2)(d_2^{l-1} \star b_2^{l-1}) \overset{(4)}{\sim} d_2^2(b_2x_2d_2)(d_2^{l-3} \star b_2^{l-3})b_2^2 \overset{(39')}{\sim} d_2^2(d_2^{l-3} \star b_2^{l-3})(b_2x_2d_2)b_2$
$\overset{(4)}{\sim} (d_2^{l-1} \star b_2^{l-1})(d_2x_2b_2) \overset{(24)}{=} \varphi(u_l\tau_1)$.

The remaining calculations are straightforward,

(15): $\varphi(\eta_2\xi_1) \overset{(24)}{=} (d_2a_2b_2^2)(d_2c_2) \overset{(4)}{\sim} d_2(a_2b_2c_2) \overset{(30)}{\sim} d_2b_2 \overset{(4)}{\sim} 1 \overset{(4)}{\sim} d_2b_2 \overset{(30)}{\sim} (a_2d_2c_2)b_2 \overset{(4)}{\sim} \varphi(\eta_1\xi_2)$,

(16): $\varphi(\eta_3\sigma_2\xi_1) \overset{(24),(4)}{\sim} d_2^2a_2b_2(b_2b_1)d_2d_1(b_2c_2) \overset{(26)}{\sim} d_2^2(a_2b_2)d_0d_2d_1(b_2c_2) \overset{(35)}{\sim} d_2^2d_0(a_2b_2)d_2d_1(b_2c_2)$
$\overset{(34)}{\sim} d_2(d_2d_0)(a_2b_2)d_2(b_2c_2)d_1 \overset{(4)}{\sim} d_2(d_2d_0)(a_2b_2c_2)d_1 \overset{(25)}{\sim} d_2b_1(a_2b_2c_2)d_1 \overset{(30)}{\sim} d_2b_1b_2d_1 \overset{(24)}{=} \varphi(\sigma_1^{-1})$,
$\varphi(\eta_1\sigma_2\xi_3) \overset{(4)}{\sim} a_2b_1d_2(d_1d_2)c_2b_2^2 \overset{(25)}{\sim} a_2b_1d_2(b_0c_2)b_2^2 \overset{(1)}{\sim} (a_0d_1)b_1d_2c_1b_2^2 \overset{(4)}{\sim} a_0d_2(b_1d_1)c_1b_2^2$
$\overset{(33)}{\sim} a_0d_2b_1b_2(d_1c_1)b_2 \overset{(37)}{\sim} (d_2b_1b_2d_1)(a_0c_1)b_2 \overset{(1)}{\sim} d_2b_1b_2d_1d_2b_2 \overset{(4)}{\sim} d_2b_1b_2d_1 \overset{(24)}{=} \varphi(\sigma_1^{-1})$,

$$(17):\ \varphi(\eta_3\tau_2\xi_1)\overset{(24),(4)}{\sim} d_2^2a_2b_2x_2b_2c_2\overset{(5)}{\sim} d_2^2(b_2x_2b_2)\overset{(4)}{\sim}\varphi(\tau_1)\overset{(4)}{\sim}(d_2x_2d_2)b_2^2\overset{(5)}{\sim} a_2d_2x_2d_2c_2b_2^2 \overset{(4)}{\sim}\varphi(\eta_1\tau_2\xi_3),$$

$$(18):\ \varphi(\sigma_1\xi_1)\overset{(24)}{=}(b_1d_2d_1b_2)(d_2c_2)\overset{(4)}{\sim} b_1d_2(d_1c_2)\overset{(28)}{\sim} b_1(d_2c_0)\overset{(28)}{\sim} b_1c_1\overset{(29)}{\sim} a_2b_2\overset{(24)}{=}\varphi(\xi_1),$$

$$\varphi(\eta_1\sigma_1)\overset{(4)}{\sim} a_2(b_2b_1)d_2d_1b_2\overset{(26)}{\sim}(a_2d_0)d_2d_1b_2\overset{(1)}{\sim}(a_1d_2)d_1b_2\overset{(1)}{\sim}(a_0d_1)b_2\overset{(1)}{\sim}\varphi(\eta_k),$$

$$(19):\ \varphi(\sigma_1\sigma_1^{-1})\overset{(24)}{=}(b_1d_2d_1b_2)(d_2b_1b_2d_1)\overset{(4)}{\sim}(b_1d_2d_1)(b_1b_2d_1)\overset{(4)}{\sim}1\overset{(4)}{\sim}(d_2b_1b_2d_1)(b_1d_2d_1b_2) \overset{(24)}{=}\varphi(\sigma_1^{-1}\sigma_1),$$

$$(20):\ \varphi(\sigma_2\sigma_1\sigma_2)\overset{(4)}{\sim} d_2b_1d_2d_1b_2^2b_1d_2^2d_1b_2^2\overset{(26)}{\sim} d_2(b_1d_2d_1b_2)d_0d_2^2d_1b_2^2\overset{(37)}{\sim} d_2d_0(b_1d_2d_1b_2)d_2^2d_1b_2^2$$
$$\overset{(25),(4)}{\sim} b_1^2d_2(d_1d_2)d_1b_2^2\overset{(26)}{\sim} b_1^2d_2(d_1d_2)d_1(d_0d_1)b_2\overset{(25)}{\sim} b_1^2d_2b_0d_1(d_0d_1)b_2\overset{(4)}{\sim} b_1^2(d_2b_0d_1d_0b_2)d_2d_1b_2$$
$$\overset{(40)}{\sim} b_1^2(b_0d_2d_1b_2d_0)d_2d_1b_2\overset{(25),(26)}{\sim} b_1^2(d_1d_2)d_2d_1b_2(b_2b_1)d_2d_1b_2\overset{(4)}{\sim} b_1d_2^2d_1b_2(b_2b_1)d_2d_1b_2$$
$$\overset{(4)}{\sim}\varphi(\sigma_1\sigma_2\sigma_1),$$

$$(21):\ \varphi(\tau_2\sigma_1\sigma_2)\overset{(4)}{\sim} d_2^2x_2b_2^2b_1d_2^2d_1b_2^2\overset{(26)}{\sim} d_2(d_2x_2b_2)d_0d_2^2d_1b_2^2\overset{(38)}{\sim} d_2d_0(d_2x_2b_2)d_2^2d_1b_2^2$$
$$\overset{(4)}{\sim}(d_2d_0)(d_2x_2d_2)d_1b_2^2\overset{(25)}{\sim} b_1(d_2x_2d_2)d_1b_2^2\overset{(4)}{\sim} b_1d_2^2(b_2x_2d_2)d_1b_2^2\overset{(39)}{\sim} b_1^2d_2^2d_1(b_2x_2d_2)b_2^2$$
$$\overset{(4)}{\sim}\varphi(\sigma_1\sigma_2\tau_1).$$

$$(22):\ \varphi(\tau_1\sigma_2\sigma_1)\overset{(4)}{\sim} d_2x_2(b_1d_2d_1b_2)(b_2b_1)d_2d_1b_2\overset{(26)}{\sim} d_2x_2(b_1d_2d_1b_2)d_0d_2d_1b_2$$
$$\overset{(37),(4)}{\sim} d_2(d_0b_0)x_2d_0(b_1d_2d_1^2)b_2\overset{(31)}{\sim} d_2d_0x_1(b_1d_2d_1^2)b_2\overset{(4)}{\sim} d_2d_0b_1(d_1x_1b_1)d_2d_1^2b_2$$
$$\overset{(38)}{\sim} d_2d_0b_1d_2(d_1x_1b_1)d_1^2b_2\overset{(2),(4)}{\sim} d_2d_0b_1d_2d_1(b_0x_2d_0)d_1b_2\overset{(25),(4)}{\sim} d_2d_0(b_1d_2d_1b_2)d_2b_0x_2b_2^2$$
$$\overset{(37)}{\sim} d_2(b_1d_2d_1b_2)d_0d_2b_0x_2b_2^2\overset{(25),(26)}{\sim} d_2b_1d_2d_1b_2(b_2b_1)d_2(d_1d_2)x_2b_2^2\overset{(4)}{\sim}\varphi(\sigma_2\sigma_1\tau_2),$$

$$(23):\ \varphi(\sigma_1\tau_1)\overset{(4)}{\sim} b_1d_2d_1x_2b_2\overset{(25)}{\sim}(d_2d_0)d_2d_1x_2b_2\overset{(6)}{\sim} d_2x_2(d_0d_2d_1)b_2\overset{(26)}{\sim} d_2x_2(b_2b_1)d_2d_1b_2\overset{(24)}{=}\varphi(\tau_1\sigma_1).$$

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