Basic embeddings of graphs and Dynnikov's three-page embedding method

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This note is devoted to the solution of two problems of contemporary geometric topology which have their method of solution in common — the technique of embedding graphs in a book with finitely many pages. First we prove a criterion for the basic embeddability of finite graphs in $\mathbb{R} \times T_{n,m}$, where $T_{n,m}$ is a bouquet of n segments (glued by their ends) and m circles. Second, the isotopy classification of arbitrary knotted graphs is reduced to the algebraic problem of the equality of central elements in two series of finitely presented semigroups.

1. Basic embeddings of graphs. The concept of a basic embedding was motivated by the Kolmogorov–Arnol'd solution of Hilbert's 13th problem. Let G, X, Y be arbitrary compact metric spaces. An embedding $G \subset X \times Y$ is called *basic* if for any continuous function $f: G \to \mathbb{R}$ there exist continuous functions $g: X \to \mathbb{R}$, $h: Y \to \mathbb{R}$, such that $f(x, y) = g(x) + h(y)$ for all $(x, y) \in G$. Let us describe the class of graphs whose basic embeddings we shall consider. By a *finite graph* G we mean any graph with finitely many vertices and edges. The *degree* $\deg v$ of a vertex $v \in G$ is the number of edges of G meeting at it. The graph need not be connected, but we shall exclude isolated vertices of degree 0. All graphs are taken to be non-oriented, but we allow multi-edges and loops. The *cycle number* $c(G)$ of G is by definition the number of connected components of G minus its Euler characteristic. An edge terminating in a vertex of degree 1 is called a *hanging* edge. A vertex $v \in G$ is considered to be *non-branching* if a hanging edge terminates in it, and to be *branching* otherwise. A vertex $v \in G$ is called *complex* if either it has degree $\deg v \geqslant 5$, or $\deg v = 4$ and it is branching. The *defect* $\delta(G)$ of G is the sum $\sum(\deg v_i - 2)$ over all complex vertices v_i of G. We recall that $T_{n,m}$ is a bouquet of n segments glued at their ends and m circles. The problem of characterizing the graphs that can be basically embedded in $\mathbb{R} \times T_{n,0}$ was posed in [1].

Theorem 1.1. *Fix integers $n, m \geqslant 0$ $(n + 2m \geqslant 2)$. Then a finite graph G can be basically embedded in $\mathbb{R} \times T_{n,m}$ if and only if $c(G) \leqslant m$ and either $\delta(G) < n + 2m$, or $\delta(G) = n + 2m$ and one of the complex vertices of G is non-branching.*

Theorem 1.2. *For any finite graph G there exist $n, m \geqslant 0$ such that G can be basically embedded in $\mathbb{R} \times T_{n,m}$. At the same time for any fixed pair of integers $n, m \geqslant 0$ there exists a finite graph that cannot be basically embedded in $\mathbb{R} \times T_{n,m}$.*

A family $\{G_i\}$ of graphs is said to be *prohibited* (for basic embeddings in $\mathbb{R} \times T_{n,m}$) if any finite graph G can be basically embedded in $\mathbb{R} \times T_{n,m}$ if and only if G contains no subgraphs homeomorphic to graphs in $\{G_i\}$.

Theorem 1.3. *There is an algorithm which, for any $n, m \geqslant 0$ and any finite graph G, verifies whether G is basically embeddable in $\mathbb{R} \times T_{n,m}$. This algorithm is quadratic in the number of vertices of G. For fixed n,m there exists a finite prohibited family for basic embeddings of finite graphs in $\mathbb{R} \times T_{n,m}$.*

Special cases of Theorems 1.1 and 1.3 (for $m = 0$) were proved in [2], Theorem 1.1 and Corollary 1.3. The proof of Theorem 1.1 uses the Sternfeld–Skopenkov geometric criterion for the basic embeddability of finite graphs ([3], GC, p. 33).

2. The three-page embedding method for knotted graphs. Fix an integer $n \geqslant 2$. Here we consider unoriented finite graphs with vertices of degrees from 2 to n. That is, we exclude hanging edges, but the graph may be non-connected and loops and multi-edges are allowed. Then a *knotted graph* is an embedding of a finite graph $G \subset \mathbb{R}^3$ to within an ambient isotopy.

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An *ambient isotopy* between knotted graphs $G, H \subset \mathbb{R}^3$ is a continuous family of homeomorphisms $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,1]$, such that $\varphi_0 = \mathrm{id}$ and $\varphi_1(G) = H$. Knotted graphs with this equivalence relation are sometimes called graphs with *non-rigid vertices*. If we require of an embedded graph $G \subset \mathbb{R}^3$ that some neighbourhood of each vertex lies in a plane during the whole course of the isotopy, then we get a graph with *rigid* vertices.

We shall consider a semigroup SG_n with generators $\mathbb{A}_n = \{a_i, b_i, c_i, d_i, x_{m,i} \mid i \in \mathbb{Z}_3 = \{0,1,2\},\ 3 \leqslant m \leqslant n\}$. The total number of letters in the alphabet $\mathbb{A}_n$ is $g(n) = 3(n+2)$. Let SG_n be the semigroup with generators in $\mathbb{A}_n$ and relations (1)–(10) below. We assume that the parameters m, p, q satisfy the inequalities $3 \leqslant m \leqslant n$, $2 \leqslant p \leqslant (n+1)/2$, and $2 \leqslant q \leqslant n/2$.

(1) $d_0 d_1 d_2 = 1$;
(2) $b_i d_i = d_i b_i = 1$;
(3) $a_i = a_{i+1} d_{i-1}$, $b_i = a_{i-1} c_{i+1}$, $c_i = b_{i-1} c_{i+1}$, $d_i = a_{i+1} c_{i-1}$;
(4) $x_{2p-1,i-1} = d_{i-1}^{p-1}(x_{2p-1,i} d_{i+1}) b_{i-1}^{p-2}$, $x_{2q,i-1} = d_{i-1}^{q-2}(b_{i+1} x_{2q,i} d_{i+1}) b_{i-1}^{q-2}$;
(5) $x_{2p-1,i} d_i^{p-1} = a_i (x_{2p-1,i} d_i^{p-1}) c_i$, $b_i^{p-1} x_{2p-1,i} b_i = a_i (b_i^{p-1} x_{2p-1,i} b_i) c_i$;
(6) $d_i x_{2q,i} d_i^{q-1} = a_i (d_i x_{2q,i} d_i^{q-1}) c_i$, $b_i^{p-1} x_{2q,i} b_i = a_i (b_i^{q-1} x_{2q,i} b_i) c_i$;
(7) $(d_i c_i) w = w (d_i c_i)$ for $w \in \{c_{i+1}, b_i d_{i+1} d_i, x_{m,i+1}\}$;
(8) $uv = vu$ for $u \in \{a_i b_i, b_{i-1} d_i d_{i-1} b_i, x_{2p-1,i} b_i, d_i x_{2q,i} b_i\}$,
$v \in \{a_{i+1}, b_{i+1}, c_{i+1}, b_i d_{i+1} d_i, x_{m,i+1}\}$;
(9) $(x_{2p-1,i} b_i) D_{p,i} = D_{p-1,i}(x_{2p-1,i} b_i)$, where $D_{k,i} = d_i^k d_{i+1}^k d_{i-1}^k$, $k \geqslant 1$;
(10) $(d_i x_{2q,i} b_i) D_{q,i} = D_{q,i}(d_i x_{2q,i} b_i)$.

One of the relations (2) is redundant: it can be obtained from relation (1) and the remaining relations in (2). Then the total number of relations in (1)–(10) is $r(n) = 3(n^2 + 7n - 2)$.

Theorem 2.1. *The centre of the semigroup SG_n classifies all knotted graphs with rigid vertices of degree $\leqslant n$ to within an ambient isotopy in $\mathbb{R}^3$.*

a) *Any knotted graph $G \subset \mathbb{R}^3$ with rigid vertices of degree $\leqslant n$ can be encoded by a certain element w_G of SG_n.*

b) *Two knotted graphs $G, H \subset \mathbb{R}^3$ with rigid vertices of degree $\leqslant n$ are isotopic if and only if we have $w_G = w_H$ in SG_n.*

c) *An element $w \in SG_n$ encodes some knotted graph with rigid vertices of degree $\leqslant n$ if and only if w is central in SG_n.*

A special case of Theorem 2.1 (with $n = 3$) was proved in [4] by the method of [5]. We replace relations (9)–(10) by (9′) $x_{m,i}(d_{i+1} d_i d_{i-1}) = x_{m,i}$.

Let SG'_n denote the semigroup with generators in $\mathbb{A}_n$ and relations (1)–(8), (9′). The new semigroup has the same number of generators and relations as SG_n.

Theorem 2.2. *The centre of the semigroup SG'_n classifies all knotted graphs with non-rigid vertices of degree $\leqslant n$ to within an ambient isotopy in $\mathbb{R}^3$.*

The proofs of Theorems 2.1 and 2.2 use Dynnikov's three-page approach [5]. We consider the following transformation of words in the alphabet $\mathbb{A}_n$: $\rho_n(a_i) = c_i$, $\rho_n(b_i) = d_i$, $\rho_n(c_i) = a_i$, $\rho_n(d_i) = b_i$, $\rho_n(x_{2p-1,i}) = x_{2p-1,i} b_i c_i$, $\rho_n(x_{2q,i}) = x_{2q,i}$. The map ρ_n can be extended by the formula $\rho_n(uv) = \rho_n(v)\rho_n(u)$ to involutory anti-automorphisms $\rho_n : SG_n \to SG_n$ and $\rho'_n : SG'_n \to SG'_n$. Let $w_G \in SG_n$ (respectively, $w'_G \in SG'_n$) be the classifying element for the knotted graph $G \subset \mathbb{R}^3$ in Theorem 2.1 (respectively, Theorem 2.2).

Corollary 2.1. *A knotted graph $G \subset \mathbb{R}^3$ with rigid* (*respectively, non-rigid*) *vertices of degree $\leqslant n$ is isotopic to its mirror image if and only if $w_G = \rho_n(w_G)$* (*respectively, $w'_G = \rho'_n(w'_G)$*) *in SG_n* (*respectively, SG'_n*)

Bibliography

[1] A. Cavicchioli, D. Repovš, and A. B. Skopenkov, *Topology Appl.* **84** (1998), 207–226.
[2] V. A. Kurlin, *Topology Appl.* **102** (2000), 113–137.
[3] A. Skopenkov, *Topology Appl.* **65** (1995), 29–48.
[4] V. A. Kurlin, *Funktsional. Anal. i Prilozhen.* **35**:3 (2001), 84–88; English transl., *Funct. Anal. Appl.* **35** (2001), 230–233.

[5] I. A. Dynnikov, *Funktsional. Anal. i Prilozhen.* **33**:4 (1999), 25–37; **34**:1 (2000), 29–40; English transl., *Funct. Anal. Appl.* **33** (1999), 260–269; **34** (2000), 24–32.

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