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PERIPHERALLY SPECIFIED HOMOMORPHS OF LINK GROUPS

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ABSTRACT

Johnson and Livingston have characterized peripheral structures in homomorphs of knot groups. We extend their approach to the case of links. The main result is an algebraic characterization of all possible peripheral structures in certain homomorphic images of link groups.

Keywords: Link; link group; longitude; meridian; Pontryagin product; Johnson–Livingston product.

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1. Introduction

1.1. *Motivation and summary of results*

Groups that can appear as the image under a surjective homomorphism of the group of a knot have been investigated by various authors, see, for instance [3–6]. Given such a homomorphic image G, it is of interest to characterize the subgroup which is the image of the peripheral subgroup of the knot.

Johnson and Livingston [6] have given necessary and sufficient conditions on elements μ and λ of G for them to be the image of the meridian, respectively the preferred longitude of the knot. These conditions involve a Pontryagin product $\langle\mu, \lambda\rangle$ in the homology group $H_2(G)$ and a Johnson–Livingston product $\{\mu, \lambda\}$ in a quotient of $H_3(G/G')$. The Pontryagin product was first used as an obstruction for the realization problem by Edmonds and Livingston in [3].

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We extend the Johnson–Livingston method to the case of r-component links. In this context we consider systems of elements $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ which are the images of the meridians, respectively the longitudes of the components of the link. We show that provided the μ_i are conjugate in G one can define an extended Johnson–Livingston product $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\}$ and give necessary and sufficient conditions for the realizability of these systems.

1.2. *Preliminary definitions*

Definition 1.1 (knots, ribbon links, ambient isotopy).
(a) A *link* L is a smooth embedding of disjoint oriented circles in S^3. The ith circle is called the *ith component* of the link L and is denoted by L_i. If $r = 1$, then the link L is *a knot*.

A link $L \subset S^3$ is *ribbon* if L bounds a disjoint union of disks $\sqcup_{i=1}^r D_i^2$ immersed into S^3, whose singularities are always as in Fig. 1 (see Sec. 2.2).

(b) Two links L and L' are *equivalent* if there is an orientation preserving self homeomorphism of S^3 sending L_i to L'_i for $i = 1, \ldots, r$ and respecting the orientations of the components. Links will be studied up to equivalence.

Definition 1.2 (meridians, longitudes, preferred systems *of longitudes*).
(a) Let $L = \cup_{i=1}^r L_i \subset S^3$ be a r-component link, $T(L_i)$ be a sufficiently small tubular neighborhood of L_i, and p_i a point on the boundary $\partial T(L_i)$, $i = 1, \ldots, r$.

A meridian m_i of the component L_i is an oriented simple closed curve $m_i \subset \partial T(L_i)$ that bounds inside $T(L_i)$ a 2-dimensional disk intersecting L_i in a single point with positive sign. The homotopy class of m_i is unique in $\pi_1(\partial T(L_i), p_i)$ and called *the meridian* of L_i.

(b) *A longitude* of the component L_i is an oriented curve $l_i \subset \partial T(L_i)$ passing through p_i and isotopic to L_i inside $T(L_i)$. A longitude of L_i is *preferred* and is denoted by $\bar{l}_i$, if there is an oriented surface $F_i \subset S^3 - \operatorname{Int} T(L_i)$ with $\partial F_i = \bar{l}_i$. The homotopy class of $\bar{l}_i$ is unique in $\pi_1(\partial T(L_i), p_i)$ and called *the preferred longitude*.

(c) Denote by $T(L)$ the disjoint union $\sqcup_{i=1}^r T(L_i)$ of sufficiently small tubular neighborhoods of $L_1, \ldots, L_r$. A system of curves $\overline{(l_1, \ldots, l_r)} \subset S^3 - L$ is *a preferred system of longitudes for* L, if the curve l_i is a longitude of the component L_i for

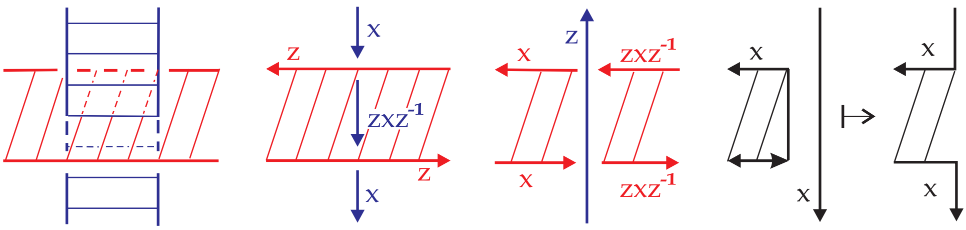

Fig. 1. A part of a ribbon link, represented bands and a band connection.

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each $i = 1, \ldots, r$, and the union $l_1 \cup \cdots \cup l_r$ is the boundary of an oriented surface $F \subset S^3 - \operatorname{Int} T(L)$.

Defnition 1.2(c) provides a natural extension to links of the notion of a preferred longitude for knots. Note that $\overline{(l_1, \ldots, l_r)} \neq (\bar{l}_1, \ldots, \bar{l}_r)$ in general. The definition of preferred longitudes and systems will be reformulated in Lemma 2.2.

We fix a system of arcs γ_i, $i = 2, \ldots, r$, properly embedded in $S^3 - \operatorname{Int} T(L)$ joining p_1 to p_i and intersecting only at p_1. We define γ_1 to be the constant path at p_1. To each oriented simple closed curve c in $\partial T(L_i)$ passing through p_i, $i = 1, \ldots, r$, we can associate the homotopy class of the loop $\gamma_i \circ c \circ \gamma_i^{-1}$. We still denote this homotopy class by c and consider it as an element of $\pi_1(S^3 - L) := \pi_1(S^3 - L, p_1)$. This holds in particular for the meridians and longitudes of L.

Definition 1.3 (a meridional system for G, realizable systems for (G, $\boldsymbol{\mu}$)). (a) Let G be a finitely generated group. Denote by G^r the direct product $G \times \cdots \times G$ (r times). A system of elements $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in G^r$ is called *meridional* for G if the group G is generated by finitely many conjugates of the elements $\mu_1, \ldots, \mu_r$.

(b) Suppose that a group G has a meridional system $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in G^r$. A system of elements $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in G^r$ is called *weakly realizable* for the pair $(G, \boldsymbol{\mu})$ if there exists a link $L = \cup_{i=1}^r L_i \subset S^3$ with a surjective homomorphism $\rho : \pi_1(S^3 - L) \to G$ such that $\rho(m_i) = \mu_i$, $\rho(l_i) = \lambda_i$ for each $i = 1, \ldots, r$, where m_i is the meridian of the component L_i, and l_i is a longitude of L_i.

(c) A weakly realizable system $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in G^r$ is *realizable* for the pair $(G, \boldsymbol{\mu})$ if $(l_1, \ldots, l_r)$ from (b) is a preferred system of longitudes for the link L. Denote by $R(G, \boldsymbol{\mu}) \subset G^r$ the set of all realizable systems $\boldsymbol{\lambda} \in G^r$ for $(G, \boldsymbol{\mu})$.

(d) Let G be a group with a meridional system $\boldsymbol{\mu} \in G^r$. Denote by G' the commutator subgroup of G. Let $\mathrm{pr} : G \to G/G'$ be the quotient map. Denote by $[\mu_1], \ldots, [\mu_r]$ the images of the meridians $\mu_1, \ldots, \mu_r \in G$ in *the abelianization* G/G'.

1.3. *Main results*

Let G be a finitely generated group. Suppose that there exist a r-component link $L \subset S^3$ and a surjective homomorphism $\rho : \pi_1(S^3 - L) \to G$. For any link $L \subset S^3$, the group $\pi_1(S^3 - L)$ has a well-known Wirtinger presentation [2] and hence a meridional system. Then G also has a meridional system obtained by selecting one Wirtinger generator m_i for each component L_i. Their images $\mu_i = \rho(m_i)$ form a meridional system $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in G^r$.

The converse was proved by González–Acuña for links [4] by using a 4-dimensional technique. We shall need the fact that every meridional system is realized by a link where all the linking numbers are zero, so we extend to ribbon links the simple geometric proof given for knots in [5, Proposition 2.3]. This result shows that the set $R(G, \boldsymbol{\mu})$ of all realizable systems is not empty.

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Theorem 1.4. *Let a group G have a meridional system $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in G^r$.*
A system $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in G^r$ is weakly realizable *for $(G, \boldsymbol{\mu})$ if and only if*

(i) *the element λ_i commutes with μ_i for each $i = 1, \ldots, r$;*
(ii) *the sum of the Pontryagin products $\sum_{i=1}^{r} \langle \mu_i, \lambda_i \rangle$ vanishes in the group $H_2(G)$.*

Let $Q(G)$ denote the quotient group $H_3(G/G')/pr_*(H_3(G))$.

Theorem 1.5. *Let a group G have a meridional system $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in G^r$ such that each μ_i is conjugate to μ_1, $i = 2, \ldots, r$.*
A system of elements $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in G^r$ is realizable *for the pair $(G, \boldsymbol{\mu})$ if and only if conditions* (i), (ii) *of Theorem* 1.4 *hold and*

(iii) $\boldsymbol{\lambda} \in (G')^r$;
(iv) *the extended Johnson–Livingston product $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\} = 0$ in $Q(G)$.*

To extend the Johnson–Livingston method we use multi-connected sums of various geometric objects such as links, surfaces and manifolds, see Definitions 4.5–4.10. The key points of the proof are the well-definedness and additivity of the extended Johnson–Livingston product, see Theorem 4.17.

1.4. *Organization of the paper*

The realizability of meridional systems is proved in Sec. 2, Proposition 2.3. Section 3 contains Definition 3.2 of the Pontryagin product and the proof of Theorem 1.4. In Sec. 4 we introduce Johnson–Livingston products in Definitions 4.1, 4.4 and prove their well-definedness. The proof of Theorem 1.5 will be finished in Sec. 5. We give examples of applications of Theorem 1.5 in Sec. 6.

2. Preferred Longitudes and Meridional Systems

Section 2.1 discusses preferred systems of longitudes. In Sec. 2.2 the realizability of meridional systems is proved using geometric operations on algebraic representations.

2.1. *Preferred systems of longitudes*

Definition 2.1 (the linking number lk, algebraically split links).
(a) Let J and K be two disjoint oriented simple closed curves in S^3. We denote by $\mathrm{lk}(J, K)$ their linking number.

(b) Let $L = \cup_{i=1}^{r} L_i \subset S^3$ be an oriented r-component link. If all the linking numbers $\mathrm{lk}(L_i, L_j) = 0$ for $i, j = 1, \ldots, r$, $i \neq j$, then the link L is called *algebraically split.*

Recall that preferred longitudes and systems were introduced in Definition 1.2.

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Lemma 2.2. *Let* $L = \sqcup_{i=1}^{r} L_i \subset S^3$ *be an oriented* r*-component link. Let* $[m_i], [l_i] \in H_1(S^3 - L)$ *be the classes of the meridian and a longitude of* L_i, $i = 1, \ldots, r$. *The system of curves* $(l_1, \ldots, l_r)$ *is preferred for the link* L *if and only if* $[l_i] = \sum_{i=1}^{r} \alpha_{ij}[m_j]$, *where* $\alpha_{ij} = \mathrm{lk}(L_i, L_j)$, $i \neq j$, $\alpha_{ii} = -\sum_{j \neq i} \mathrm{lk}(L_i, L_j)$ *for all* $i, j = 1, \ldots, r$.

Proof. The homology group $H_1(S^3 - L)$ is isomorphic to $\mathbb{Z}[m_1] \oplus \cdots \oplus \mathbb{Z}[m_r]$, where m_i is the meridian of the component L_i, $i = 1, \ldots, r$. By Definition 1.2(c) a system $(l_1, \ldots, l_r)$ of longitudes is preferred if there is an oriented surface $F \subset S^3 - \mathrm{Int}\, T(L)$ with boundary $\partial F = \sqcup_{i=1}^{r} l_i$.

Hence in the group $H_1(S^3 - L)$ one has $\sum_{i=1}^{r} [l_i] = 0$. The preferred longitude $\bar{l}_i$ is the boundary of an oriented surface $F_i \subset S^3 - \mathrm{Int}\, T(L_i)$, $i = 1, \ldots, r$. The surface F_i can be used to compute the linking number $\alpha_{ij} = \mathrm{lk}(L_i, L_j)$, $i \neq j$. Then $[\bar{l}_i] = \sum_{j \neq i} \alpha_{ij}[m_j]$ in $H_1(S^3 - L)$.

By Definition 1.2(b) any longitude $l_i \subset \partial T(L_i)$ is isotopic inside $T(L_i)$ to L_i, hence $l_i = m_i^{\alpha_{ii}} \bar{l}_i$ in $\pi_1(\partial T(L_i))$ for some $\alpha_{ii} \in \mathbb{Z}$, $i = 1, \ldots, r$. Equivalently, in the homology group $H_1(S^3 - L)$ one gets $[l_i] = \alpha_{ii}[m_i] + [\bar{l}_i] = \sum_{j=1}^{r} \alpha_{ij}[m_j]$. The condition $\sum_{i=1}^{r} [l_i] = 0$ is equivalent to $\alpha_{ii} = -\sum_{j \neq i} \alpha_{ij}$ as required.

Conversely, construct a smooth map $f : \sqcup \partial T(L_i) \to S^1$ such that the restriction to the meridians m_i of L_i is a degree one map and f is constant on the curves l_i. The only obstructions to the extension of the map f to a smooth map $\tilde{f} : S^3 - \sqcup_{i=1}^{r} \mathrm{Int}\, T(L_i) \to S^1$ are the conditions $\alpha_{ii} + \sum_{j \neq i} \alpha_{ij} = 0$, $i = 1, \ldots, r$.

An inverse image of a regular value of the extended map $\tilde{f}$ gives an oriented surface $F \subset S^3 - \sqcup_{i=1}^{r} \mathrm{Int}\, T(L_i)$ such that the boundary $\partial F = l_1 \cup \cdots \cup l_r$. Hence the system of curves $(l_1, \ldots, l_r)$ is preferred for the link L. □

2.2. *Realizability of meridional systems*

Proposition 2.3. *Suppose that a group* G *has a meridional system* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in G^r$. *Then there exists a ribbon* r*-component link* $L \subset S^3$ *with a surjective homomorphism* $\rho : \pi_1(S^3 - L) \to G$ *such that* $\rho(m_i) = \mu_i$ *for each* $i = 1, \ldots, r$, *where* m_i *is the meridian of the component* L_i.

The proof of Proposition 2.3 is a straightforward generalization of Johnson's proof for knots. We refer the reader to [5] for details. Recall that to describe a homomorphism from the link group to the group G, it suffices to label the arcs of a planar diagram of the link with elements of G in such a way that, at each crossing, the corresponding relation for the labelled elements holds in G.

A represented band is a pair of parallel, oppositely directed arcs of the diagram of the link, with no other arcs of the link passing between the two arcs of the band and such that the two arcs are labelled with the same element of G, see Fig. 1.

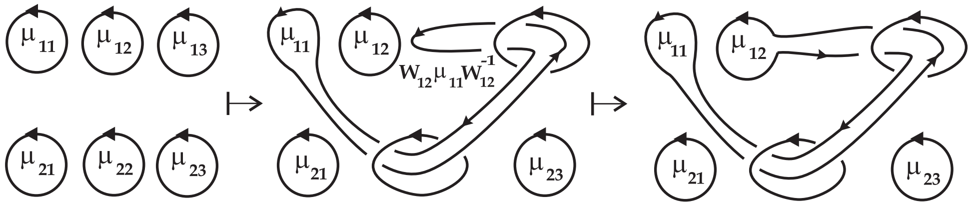

Fig. 2. The construction of a ribbon link L with a homomorphism $\rho : \pi_1(S^3 - L) \to G$.

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Proof of Proposition 2.3. Let $\mu_1, \ldots, \mu_r$ be a meridional system for G. There are positive integers $k_1, \ldots, k_r$ and $\mu_{ij} \in G$, $i = 1, \ldots, r$, $j = 1, \ldots, k_i$ such that:

- $\mu_{i1} = \mu_i$ for all $i = 1, \ldots, r$;
- the μ_{ij} generate the group G;
- $\mu_{ij} = W_{ij}\mu_i W_{ij}^{-1}$ for words W_{ij} in the letters μ_{uv}, $i = 1, \ldots, r$, $j = 1, \ldots, k_i$.

Consider the trivial link of $k_1 + \cdots + k_r$ components and label the components with the μ_{ij} (see Fig. 2). This gives a surjective homomorphism from the group of the trivial link to G. For each pair of i, j weave a represented band issuing from the component labelled with μ_{i1} according to the instructions given by W_{ij}.

Perform a band connection (see Fig. 1) between the represented band and the component labelled with μ_{ij}. One obtains in this way a ribbon link L of r components and a surjective homomorphism $\rho : \pi_1(S^3 - L) \to G$ (see Fig. 2). □

3. Weakly Realizable Systems: Proof of Theorem 1.4

In Sec. 3.1 we introduce the Pontryagin product. Section 3.2 is devoted to necessity in Theorem 1.4. Section 3.3 contains the proof of sufficiency in Theorem 1.4.

3.1. *Pontryagin product*

Definition 3.1 (the G-bordism group $\Omega_n(G)$ of *n-dimensional manifolds*). (a) Let M, N be two oriented closed n-dimensional possibly disconnected manifolds and let G be a group. Fix homotopy classes of continuous maps $f_M : M \to K(G, 1)$, $f_N : N \to K(G, 1)$.

The pairs (M, f_M) and (N, f_N) are called *G-bordant* if there is an oriented compact connected $(n+1)$-dimensional manifold W with a continuous map $f_W : W \to K(G, 1)$ such that $\partial W = M \cup (-N)$, $f_W|_M = f_M$ and $f_W|_N = f_N$. Here $(-N)$ is N with the reversed orientation.

(b) Classes of G-bordant pairs $[M, f_M]$ form *the G-bordism group* $\Omega_n(G)$. The operation is the disjoint union, the unit element is the empty set $\varnothing$ with the empty map $\varnothing \to K(G, 1)$.

(c) Denote by $H_n(G)$ the nth homology group of a group G with integer coefficients [1]. It is a well-known fact, which can be proved using the Atiyah–Hirzebruch spectral sequence (see [9, Theorem 15.7]), that for $n = 2$ and 3, the natural map $\iota_n : \Omega_n(G) \to H_n(G)$ is a group isomorphism.

Definition 3.2 (the Pontryagin product $\langle \boldsymbol{\mu}, \boldsymbol{\lambda} \rangle$ in the group $\mathbf{H_2(G)}$.)
(a) Suppose that two elements $\mu, \lambda \in G$ commute. Then there is a natural homomorphism $\psi : \mathbb{Z} \times \mathbb{Z} \to G$ defined by $\psi(1,0) = \mu$, $\psi(0,1) = \lambda$. The homomorphism ψ induces a representation $\rho : \pi_1(S^1 \times S^1) \to G$ and a continuous map $f : S^1 \times S^1 \to K(G,1)$. So, we get an element $[S^1 \times S^1, f] \in \Omega_2(G) \cong H_2(G)$.

(b) The element $\langle \mu, \lambda \rangle = \iota_2([S^1 \times S^1, f]) \in H_2(G)$ is called *the Pontryagin product*. It is well-defined and satisfies the relation $\langle \mu, \lambda_1 \rangle + \langle \mu, \lambda_2 \rangle = \langle \mu, \lambda_1 \lambda_2 \rangle$ when λ_1 and λ_2 both commute with μ in G, see [1].

We shall use repeatedly the following identification: let W be a connected manifold with some base point x_0. There is a one-to-one correspondence between representations $\rho : \pi_1(W, x_0) \to G$ and homotopy classes of continuous pointed maps $f : W \to K(G,1)$, see [9, Chap. 6, Theorem 6.39(ii)].

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3.2. *Necessity in Theorem 1.4*

Lemma 3.3. *Let G be a group with a meridional system $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in G^r$. If a system $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in G^r$ is weakly realizable for the pair $(G, \boldsymbol{\mu})$, then condition* (i) *of Theorem* 1.4 *holds.*

Proof. Let curves m_i, l_i be associated to the elements μ_i, λ_i by Definition 1.3(b), $i = 1, \ldots, r$. Then m_i, l_i lie on the boundary of a small tubular neighborhood $T(L_i)$ of $L_i \subset L$. Since $\pi_1(\partial T(L_i)) \cong \mathbb{Z} \oplus \mathbb{Z}$, then the corresponding loops $m_i, l_i \in \pi_1(S^3 - L)$ commute. Hence their images $\rho(m_i) = \mu_i$, $\rho(l_i) = \lambda_i$ commute in G. □

Lemma 3.4. *Under the conditions of Theorem* 1.4, *for each $i = 1, \ldots, r$, fix an element $\lambda_i \in G$ commuting with μ_i. Suppose that there exists an oriented compact* connected 3-*manifold M with a representation $\rho : \pi_1(M) \to G$ such that*

$$\partial M = \sqcup_{i=1}^r (S_i^1 \times S_i^1),\ \rho|_{\partial M}(\{\text{pt}\} \times S_i^1) = \mu_i,\ \rho|_{\partial M}(S_i^1 \times \{\text{pt}\}) = \lambda_i,\ i = 1, \ldots, r.$$

Then the sum of the Pontryagin products $\sum_{i=1}^r \langle \mu_i, \lambda_i \rangle$ vanishes in the group $H_2(G)$.

Proof. Let $f : M \to K(G,1)$ be a continuous map corresponding to the homomorphism $\rho : \pi_1(M) \to G$.

We have $\iota_2([\partial M, f|_{\partial M}]) = \sum_{i=1}^r \iota_2([S_i^1 \times S_i^1, f|_{S_i^1 \times S_i^1}]) = \sum_{i=1}^r \langle \mu_i, \lambda_i \rangle$ in the group $H_2(G)$. Since the disjoint union $\sqcup_{i=1}^r (S_i^1 \times S_i^1)$ is bounded by an oriented compact connected 3-manifold M, then $[\partial M, f|_{\partial M}] = 0$, i.e. $\sum_{i=1}^r \langle \mu_i, \lambda_i \rangle = 0$. □

Lemma 3.5. *Let G be a group with a meridional system $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in G^r$. If a system $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in G^r$ is weakly realizable for the pair $(G, \boldsymbol{\mu})$, then condition* (ii) *of Theorem* 1.4 *holds.*

Proof. Let $L \subset S^3$ be a link weakly realizing the given system $\boldsymbol{\lambda} \in G^r$, see Definition 1.3(b). Let $T(L) \subset S^3$ be a sufficiently small tubular neighborhood of L. Then Lemma 3.5 follows from Lemma 3.4 for $M = S^3 - \operatorname{Int} T(L)$. $\square$

Necessity in Theorem 1.4 follows directly from Lemmas 3.3 and 3.5.

3.3. *Sufficiency in Theorem 1.4*

Lemmas 3.3 and 3.5 motivate the following definition.

Definition 3.6 (an algebraic triple $(\mathbf{G}, \boldsymbol{\mu}, \boldsymbol{\lambda})$).
Let G be a group with a meridional system $\boldsymbol{\mu} \in G^r$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in G^r$ be a system such that conditions (i) and (ii) of Theorem 1.4 hold. Then $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$ is called *an algebraic triple*.

Let G be a group with a meridional system $\boldsymbol{\mu} \in G^r$. By Definition 3.6 and Lemmas 3.3, 3.5 if a system $\boldsymbol{\lambda} \in G^r$ is weakly realizable for the pair $(G, \boldsymbol{\mu})$, then the triple $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$ is algebraic. Our purpose is to prove the converse.

Definition 3.7 (a geometric triple $(\mathbf{M}, \boldsymbol{\rho}, \mathbf{f})$).
Let M be a connected oriented compact 3-manifold with $\partial M = \sqcup_{i=1}^r (S_i^1 \times S_i^1)$. Suppose that there is a surjective homomorphism $\rho : \pi_1(M) \to G$ such that

- for *the meridian* $m_i = \{\text{pt}\} \times S_i^1 \subset \partial M$, we have $\rho(m_i) = \mu_i$;
- for *the longitude* $l_i = S_i^1 \times \{\text{pt}\} \subset \partial M$, we have $\rho(l_i) = \lambda_i$ for each $i = 1, \ldots, r$.

Take a continuous map $f : M \to K(G, 1)$ associated to ρ. Then (M, ρ, f) is said to be *a geometric triple* corresponding to the algebraic triple $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$.

Lemma 3.8. *For any algebraic triple* $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$, *there is a corresponding geometric triple* (M, ρ, f).

Proof. By Definition 3.6 the Pontryagin products $\langle \mu_i, \lambda_i \rangle \in H_2(G)$ are well-defined. We have representations $\rho|_{S_i^1 \times S_i^1} : \pi_1(S_i^1 \times S_i^1) \to G$. There are continuous maps $f_i : S_i^1 \times S_i^1 \to K(G, 1)$ such that $\iota_2([S_i^1 \times S_i^1, f_i]) = \langle \mu_i, \lambda_i \rangle$.

By condition (ii) of Theorem 1.4 the element $[\sqcup_{i=1}^r (S_i^1 \times S_i^1), \sqcup_{i=1}^r f_i] = \iota_2^{-1}(\sum_{i=1}^r \langle \mu_i, \lambda_i \rangle)$ vanishes in $\Omega_2(G)$. By Definition 3.1(a) there are a 3-manifold M and a continuous map $f : M \to K(G, 1)$ extending the maps f_i. We can, if necessary, add 1-handles to M to make it connected and add connected sums of $S^2 \times S^1$ to make the homomorphism corresponding to f surjective. $\square$

Lemma 3.9. *Let* $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$ *be an algebraic triple and let* (M, ρ, f) *be a corresponding geometric triple. Denote by* W *the closed* 3-*manifold* $M \cup (S^1 \times (\sqcup_{i=1}^r D_i^2))$, *where*

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$\{\mathrm{pt}\} \times \partial D_i^2$ *is glued to* $\{\mathrm{pt}\} \times S_i^1$ *in* ∂M. *Fix a point* $q_i \in \partial D_i^2$, $i = 1, \ldots, r$.

(a) *Any closed curve* $\gamma \subset M$ *is isotopic, in* W, *to a curve* $\gamma' \subset M$ *with* $\rho(\gamma') = e$.
(b) *There exists an integral surgery carrying* W *to* S^3 *in such a way that*

- $S^1 \times \cup_{i=1}^r \{q_i\} \subset S^1 \times (\sqcup_{i=1}^r D_i^2) \subset W$ *maps to a link* $L' = \cup_{i=1}^r L_i' \subset S^3$;
- $\rho : \pi_1(M) \to G$ *turns into a surjective homomorphism* $\rho' : \pi_1(S^3 - L') \to G$;
- *for the meridian* m_i' *and a longitude* l_i' *of* L_i', *one has* $\rho'(m_i') = \mu_i$, $\rho'(l_i') = \lambda_i \mu_i^{a_i}$ *for some integer* a_i.

Proof. The proof of (a) is completely analogous to [6, Claim on p. 140].

(b) Any oriented closed 3-manifold W can be obtained from S^3 by an integral surgery [7, Chap. 9, Sec. I]. Using (a) we can perform such a surgery along curves $\gamma \subset M$ with $\rho(\gamma) = e$. Hence we get a link $L' \subset S^3$ and a surjective homomorphism $\rho' : \pi_1(S^3 - L') \to G$. The meridian m_i always maps to the meridian m_i', i.e. $\rho'(m_i') = \rho(m_i) = \mu_i$. But the longitude l_i maps to some longitude $l_i' m_i^{-a_i}$, hence $\rho'(l_i') = \lambda_i \mu_i^{a_i}$, where $a_i \in \mathbb{Z}$. □

Proposition 3.10. *Let* G *be a group with a meridional system* $\boldsymbol{\mu} \in G^r$. *A system* $\boldsymbol{\lambda}$ *is weakly realizable for* $(G, \boldsymbol{\mu})$ *if the triple* $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$ *is algebraic.*

Proof. Suppose that $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$ is an algebraic triple. Let (M, ρ, f) be a corresponding geometric triple from Lemma 3.8. By forming the connected sum of M with copies of $S^1 \times S^2$ we add free generators to $\pi_1(M)$ and hence can arrange that the representation $\rho : \pi_1(M) \to G$ is surjective. Apply Lemma 3.9(b) to the manifold M and the homomorphism ρ. □

Sufficiency in Theorem 1.4 follows directly from Proposition 3.10.

4. Johnson–Livingston Products

4.1. *Definitions of Johnson–Livingston products*

Firstly, we introduce the usual Johnson–Livingston product in Definition 4.1(c). The extended Johnson-Livingston product will appear in Definition 4.4.

Definition 4.1 (the Johnson–Livingston product $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\} \in \mathbf{Q(G)}$).
(a) Suppose that the Pontryagin product $\langle \mu, \lambda \rangle$ of two commuting elements $\mu, \lambda \in G$ vanishes in $H_2(G)$. Since $H_2(G) \cong \Omega_2(G)$, there is an oriented compact connected 3-manifold M with a surjective homomorphism $\rho : \pi_1(M) \to G$ such that

$$\partial M = S^1 \times S^1, \quad \rho|_{\partial M}(\{\mathrm{pt}\} \times S^1) = \mu \quad \text{and} \quad \rho|_{\partial M}(S^1 \times \{\mathrm{pt}\}) = \lambda.$$

(b) Suppose that $\lambda \in G'$. Form the closed manifold $U = M \cup (D^2 \times S^1)$, where ∂M is identified with $\partial(D^2 \times S^1)$ in such a way that the meridian $\partial D^2 \times \{\mathrm{pt}\} \subset D^2 \times S^1$

is glued to $S^1 \times \{\mathrm{pt}\} \subset \partial M$. This gluing kills the longitude $S^1 \times \{\mathrm{pt}\}$, which is in the kernel of $\hat{\rho} = \mathrm{pr} \circ \rho : \pi_1(M) \to G/G'$. By the Seifert–van-Kampen theorem, $\hat{\rho} = \mathrm{pr} \circ \rho$ extends to a homomorphism $\tilde{\rho} : \pi_1(U) \to G/G'$. We get the corresponding map $\tilde{f} : U \to K(G/G', 1)$ and an element $[U, \tilde{f}] \in \Omega_3(G/G') \cong H_3(G/G')$.

(c) Let $q : H_3(G/G') \to Q(G) = H_3(G/G')/\mathrm{pr}_*(H_3(G))$ be the projection map. The element $\{\mu, \lambda\} = q \circ \iota_3([U, \tilde{f}]) \in Q(G)$ is *the Johnson–Livingston product*.

Recall that the Pontryagin product is additive [1]. Then one gets a well-defined homomorphism $\theta_\mu : Z(\mu) \to H_2(G)$, $\theta_\mu(\lambda) = \langle \mu, \lambda \rangle$. Here $Z(\mu) \subset G$ is the centralizer subgroup of $\mu \in G$. Denote by $P(G, \mu) \subset Z(\mu)$ the kernel of θ_μ.

Proposition 4.2 [**6, Sec. 1**]. *For* $\lambda \in P(G, \mu)$, *the Johnson–Livingston product* $\{\mu, \lambda\} \in Q(G)$ *is well-defined. Moreover*, $\{\mu, \lambda\} + \{\mu, \tau\} = \{\mu, \lambda \cdot \tau\}$ *when defined. Hence*, *the map* $\chi_\mu : P(G, \mu) \to Q(G)$ *is a group homomorphism.*

We introduce the notion of a geometric pentad which will be used in the definition of the extended Johnson–Livingston product below.

Definition 4.3 (a geometric pentad $(\mathbf{U}, \mathbf{F}, \mathbf{M}, \boldsymbol{\rho}, \boldsymbol{f})$).
Let $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$ be an algebraic triple, (M, ρ, f) be a corresponding geometric triple. Take a connected oriented surface F with boundary $\partial F = \sqcup_{i=1}^r S_i^1$. Let $p_i \in S_i^1 \subset \partial F$ be a point, $i = 1, \ldots, r$. Form the closed manifold $U = M \cup (F \times S^1)$, where $\partial F \times S^1$ is identified with ∂M in such a way that

- *the meridian* $m_i = \{\mathrm{pt}\} \times S_i^1 \subset \partial M$ is glued to $\{p_i\} \times S^1 \subset \partial F \times S^1$;
- *the longitude* $l_i = S_i^1 \times \{\mathrm{pt}\} \subset \partial M$ is glued to $S_i^1 \times \{\mathrm{pt}\} \subset \partial F \times S^1$

for each $i = 1, \ldots, r$. The pentad (U, F, M, ρ, f) with all the above properties is called *a geometric pentad* corresponding to the algebraic triple $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$.

Definition 4.4 (the extended Johnson–Livingston product $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\} \in \boldsymbol{Q(G)}$).
Let $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$ be an algebraic triple. Suppose that the elements μ_i of the meridional system $\boldsymbol{\mu}$ are conjugate to one another in G. Assume that $\boldsymbol{\lambda} \in (G')^r$. Let (U, F, M, ρ, f) be a corresponding geometric pentad. We shall show in Lemma 4.13 that the homomorphism $\hat{\rho} = \mathrm{pr} \circ \rho : \pi_1(M) \to G/G'$ extends to a homomorphism $\tilde{\rho} : \pi_1(U) \to G/G'$. We get the corresponding map $\tilde{f} : U \to K(G/G', 1)$ and an element $[U, \tilde{f}] \in \Omega_3(G/G')$. Its image $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\} = q \circ \iota_3([U, \tilde{f}])$ in $Q(G)$ is well-defined and is called *the extended Johnson–Livingston product*, see Theorem 4.17.

In [6, Appendix 3], there is an example of a group G such that $Q(G)$ is non-trivial.

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4.2. *Multi-connected sums*

Definition 4.5 (a multi-connected sum *of manifolds*).
(a) Let $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$ and $(G, \boldsymbol{\mu}, \boldsymbol{\tau})$ be algebraic triples. Assume that condition (iii) of Theorem 1.5 holds for $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$. Let $(U_M, F_M, M, \rho_M, f_M)$ and $(U_N, F_N, N, \rho_N, f_N)$ be geometric pentads corresponding to $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$ and $(G, \boldsymbol{\mu}, \boldsymbol{\tau})$, respectively.

(b) Denote by B_i and C_i sufficiently small annular neighborhoods of the meridians $\{\text{pt}\} \times S^1_i \subset \partial M$ and $\{\text{pt}\} \times S^1_i \subset \partial N$, respectively, $i = 1, \ldots, r$. Let $A = \sqcup_{i=1}^r A_i$ be the disjoint union of r annuli A_i.

(c) Set $M\#_r N = M \cup (A \times [0,1]) \cup N$, where $A_i \times \{0\}$ is identified with $B_i \subset \partial M$, $A_i \times \{1\}$ is identified with $C_i \subset \partial N$, $i = 1, \ldots, r$. The manifold $M\#_r N$ is called *a multi-connected sum* of the manifolds M, N, see Fig. 3.

Definition 4.6 (a multi-band sum *of surfaces*).
Denote by I the disjoint union $\sqcup_{i=1}^r I_i$ of r segments. Set $b_i = B_i \cap F$, $c_i = C_i \cap F$. Set $F := F_M\#_r F_N = F_M \cup (I \times [0,1]) \cup F_N$, where $I_i \times \{0\}$ is identified with $b_i \subset \partial F_M$, $I_i \times \{1\}$ is identified with $c_i \subset \partial F_N$. The surface $F := F_M\#_r F_N$ is *a multi-band sum* of the surfaces F_M and F_N.

Definition 4.7 (a multi-connected sum *of geometric pentads*).
By using $M\#_r N$ and the surface F one can construct a closed manifold W. By Lemma 4.8(a) below the homomorphisms $\rho_M : \pi_1(M) \to G$, $\rho_N : \pi_1(N) \to G$ and the continuous maps $f_M : M \to K(G,1)$, $f_N : N \to K(G,1)$ can be extended to a homomorphism $\rho : \pi_1(M\#_r N) \to G$ and a continuous map $f : M\#_r N \to K(G,1)$. The pentad $(W, F, M\#_r N, \rho, f)$ is said to be *a multi-connected sum* of pentads.

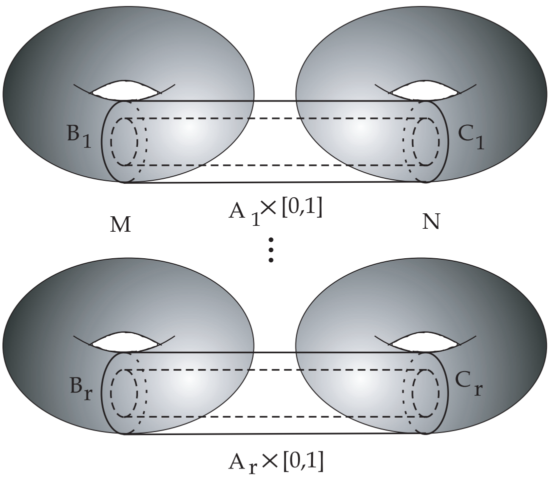

Fig. 3. A multi-connected sum $M\#_r N$ of 3-dimensional manifolds.

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Lemma 4.8. *The geometric pentad* $(W, F, M\#_r N, \rho, f)$ *corresponds to the algebraic triple* $(G, \boldsymbol{\mu}, \boldsymbol{\lambda}\cdot\boldsymbol{\tau})$. *In more details,*

(a) *the homomorphisms* $\rho_M : \pi_1(M) \to G$, $\rho_N : \pi_1(N) \to G$ *and the corresponding maps* $f_M : M \to K(G,1)$, $f_N : N \to K(G,1)$ *can be extended to a homomorphism* $\rho : \pi_1(M\#_r N) \to G$ *and a continuous map* $f : M\#_r N \to K(G,1)$;
(b) *we have* $\partial(M\#_r N) = \sqcup_{i=1}^r S_i^1 \times S_i^1$, $\rho(m_i) = \mu_i$, $\rho(l_i) = \lambda_i\tau_i$, *where* $m_i = \{\mathrm{pt}\} \times S_i^1 \subset \partial(M\#_r N)$, $l_i = S_i^1 \times \{\mathrm{pt}\} \subset \partial(M\#_r N)$, $i = 1, \ldots, r$.

Proof. A Seifert–Van-Kampen argument taking into account the arcs joining the base point to boundary components of M and N shows that ρ_M and ρ_N can be extended to ρ with the required properties, see Fig. 3. □

Definition 4.9 (a multi-connected sum *of links*).
A link $L = \cup_{i=1}^r L_i \subset S^3$ is called *a multi-connected sum* of links $K = \cup_{i=1}^r K_i \subset S^3$ and $J = \cup_{i=1}^r J_i \subset S^3$, if there exist a two-sided 2-sphere $S \subset S^3$ and arcs $I_i \subset S$, such that $K_i \cap J_i = I_i$, $(K_i \cup J_i) - I_i = L_i$ for each $i = 1, \ldots, r$, and K lies inside S, J lies outside S. In other words, we simultaneously make r usual connected sums on different components of K and J. We shall denote a multi-connected sum by $L = K\#_r J$, see Fig. 4.

If $r = 1$, then $L = K\#_1 J$ is the usual *connected sum* of knots. The following lemma is geometrically obvious.

Lemma 4.10. *Let* $L = \cup_{i=1}^r L_i$ *and* $L' = \cup_{i=1}^r L'_i$ *be links in* S^3. *If* $(l_1, \ldots, l_r)$ *and* $(l'_1, \ldots, l'_r)$ *are systems of preferred longitudes for the link* L *and* L', *respectively, then the usual connected sums* $(l_1\#_1 l'_1, \ldots, l_r\#_1 l'_r)$ *form a preferred system of longitudes for the multi-connected sum* $L\#_r L'$.

Actually, if $\partial F = \cup_{i=1}^r l_i$ and $\partial F' = \cup_{i=1}^r l'_i$, then $\partial(F\#_r F') = \cup_{i=1}^r (l_i\#_1 l'_i)$.

Proposition 4.11. (a) *The set* $R(G, \boldsymbol{\mu})$ *is a subgroup of* G^r. *The trivial element of* $R(G, \boldsymbol{\mu})$ *is represented by an algebraically split link. If links* K *and* J *realize systems* $\boldsymbol{\lambda}, \boldsymbol{\tau} \in G^r$, *respectively, then a multi-connected sum* $K\#_r J$ *realizes* $\boldsymbol{\lambda}\cdot\boldsymbol{\tau}$.

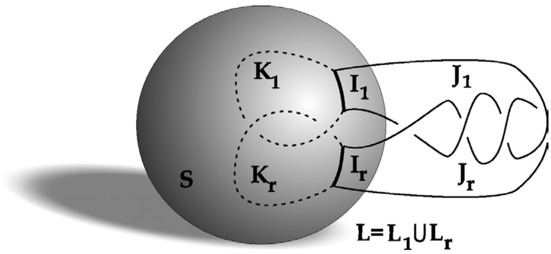

Fig. 4. A multi-connected sum $L = K\#_r N$ of links.

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(b) *Let* $(\mu_1, \ldots, \mu_r)$ *be a meridional system of a group* G *and suppose that* $(\lambda_1, \ldots, \lambda_r)$ *is realized by a link* $L \subset S^3$. *Then, for any* $\mu_{r+1} \in G$, $(\mu_1, \ldots, \mu_r, \mu_{r+1})$ *is a meridional system and* $(\lambda_1, \ldots, \lambda_r, e)$ *is realized by the link* $L \sqcup O \subset S^3$, *i.e. the link* L *with a split off trivial component added to it.*

Proof. Let $K = \cup_{i=1}^r K_i \subset S^3$ and $J = \cup_{i=1}^r J_i \subset S^3$ be two links realizing systems $\boldsymbol{\lambda}, \boldsymbol{\tau} \in R(G, \boldsymbol{\mu})$. We are going to construct a link $L = \cup_{i=1}^r L_i \subset S^3$ realizing the system $\boldsymbol{\lambda} \cdot \boldsymbol{\tau} = (\lambda_1\tau_1, \ldots, \lambda_r\tau_r)$. Take a multi-connected sum $L = K\#_r J$. Let $B, C \subset S^3$ be the internal and external 3-balls bounded by the sphere S from Definition 4.9. Then the following diagram is commutative.

$$\begin{array}{ccc}
\pi_1(S - \cup_{i=1}^r \partial I_i) & \longrightarrow & \pi_1(B - \cup_{i=1}^r (K_i - \operatorname{Int} I_i)) \\
\downarrow & & \downarrow \\
\pi_1(C - \cup_{i=1}^r (J_i - \operatorname{Int} I_i)) & \longrightarrow & \pi_1(S^3 - L)
\end{array}$$

By the Seifert–van-Kampen theorem, we may extend the given surjective representations $\rho_K : \pi_1(S^3 - K) \to G$ and $\rho_J : \pi_1(S^3 - J) \to G$ to a surjective representation $\rho_L : \pi_1(S^3 - L) \to G$ with peripheral information specified as required by Lemma 4.10.

To realize the inverse of a system $\boldsymbol{\lambda}$ in $R(G, \boldsymbol{\mu})$, realize $\boldsymbol{\lambda}$ by a link L and reverse the orientations on the link components. Take a ribbon link L which realizes some element $\boldsymbol{\lambda} \in R(G, \boldsymbol{\mu})$ by using Proposition 2.3. The multi-connected sum of L and its inverse is an algebraically split link which realizes the trivial element.

The item (b) is geometrically obvious. □

4.3. *Lifting continous maps and homomorphisms*

Lemma 4.12. *Let* H *be a group,* U_1, U_2, W *be compact* n*-dimensional manifolds with boundary. Take orientation preserving diffeomorphisms* $g_1 : \partial W \to \partial U_1$ *and* $g_2 : \partial W \to \partial U_2$. *Let* $f_1 : U_1 \to K(H, 1)$, $f_2 : U_2 \to K(H, 1)$, $f : W \to K(H, 1)$ *be continuous maps such that the following diagrams are commutative*:

$$\begin{array}{ccc}
\partial W & \xrightarrow{f|_{\partial W}} & K(H,1) \\
{\scriptstyle g_1}\downarrow & & \| \\
\partial U_1 & \xrightarrow{f_1|_{\partial U_1}} & K(H,1),
\end{array}
\qquad
\begin{array}{ccc}
\partial W & \xrightarrow{f|_{\partial W}} & K(H,1) \\
{\scriptstyle g_2}\downarrow & & \| \\
\partial U_2 & \xrightarrow{f_2|_{\partial U_2}} & K(H,1).
\end{array}$$

Then f_1, f_2 *and* f *extend to continuous maps*

$$\tilde{f}_1 : U_1 \cup_{g_1} (-W) \to K(H, 1) \quad \text{and} \quad \tilde{f}_2 : U_2 \cup_{g_2} (-W) \to K(H, 1)$$

such that in the group $\Omega_n(H)$ *we have*

$$[U_1 \cup_{g_1} (-W), \tilde{f}_1] - [U_2 \cup_{g_2} (-W), \tilde{f}_2] = [U_1 \cup_{g_3} (-U_2), \tilde{f}_3],$$

where $g_3 = g_1 \circ g_2^{-1} : \partial U_2 \to \partial W \to \partial U_1$ *and* $\tilde{f}_3$ *extends* f_1 *and* f_2.

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Proof. It is clear that the maps f_1, f_2 and f extend. Consider the $(n+1)$-dimensional manifold with corners

$$X = (U_1 \cup_{g_1} (-W)) \times [0,1] \cup_{h_1} (W \times [0,1]) \cup_{h_2} (U_2 \cup_{g_2} (-W)) \times [0,1],$$

where h_1 identifies $W \times \{0\}$ with $(-W) \times \{1\} \subset (U_1 \cup_{g_1} (-W)) \times \{1\}$ and h_2 identifies $W \times \{1\}$ with $(-W) \times \{0\} \subset (U_2 \cup_{g_2} (-W)) \times \{0\}$. The maps $\tilde{f}_1$ and $\tilde{f}_2$ extend to a map $\tilde{f} : X \to K(H,1)$.

The boundary of X has three components diffeomorphic to $U_1 \cup_{g_1} (-W)$, $(-U_2) \cup_{g_2} W$ and $(-U_1) \cup (\partial W \times [0,1]) \cup U_2$ respectively with corresponding maps to $K(H,1)$. This proves the equality in $\Omega_n(H)$ since $[X, \tilde{f}] = 0$ in $\Omega_{n+1}(H)$. □

Let $\boldsymbol{\mu}$ be a meridional system of a group G. From now on, we shall suppose that the elements μ_i of the system $\boldsymbol{\mu}$ are conjugate to one another in G.

Lemma 4.13. *Let (U, F, M, ρ_M, f_M) be a geometric pentad corresponding to an algebraic triple $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$. If $\boldsymbol{\lambda} \in (G')^r$, then the homomorphism $\hat{\rho}_M = \mathrm{pr} \circ \rho_M : \pi_1(M) \to G \to G/G'$ can be extended to a homomorphism $\tilde{\rho}_M : \pi_1(U) \to G/G'$.*

Proof. A Mayer–Vietoris sequence argument shows that it suffices to find a homomorphism $\phi_M : H_1(F \times S^1) \to G/G'$ such that the diagram is commutative:

$$\begin{array}{ccc} H_1(\partial F \times S^1) & \longrightarrow & H_1(F \times S^1) \\ \downarrow & & \downarrow {\scriptstyle \phi_M} \\ H_1(M) & \xrightarrow{\hat{\rho}_M} & G/G' \end{array}$$

One has $H_1(F \times S^1) \cong H_1(F) \otimes H_0(S^1) \oplus H_0(F) \otimes H_1(S^1)$. Define the restriction of ϕ_M to $H_1(F) \otimes H_0(S^1)$ to be the zero map. Let the generator of $H_0(F) \otimes H_1(S^1)$ map under ϕ_M to $[\mu_1]$. The diagram above commutes since $\hat{\rho}_M(l_i) = 0$ and $\hat{\rho}_M(m_i) = [\mu_1]$ as all μ_i are conjugate to one another. □

Lemma 4.14. *Under the conditions of Lemma* 4.13 *suppose that $\lambda_1 = \cdots = \lambda_r = e$ in G. Then the homomorphism $\tilde{\rho} : \pi_1(U) \to G/G'$ from Lemma* 4.13 *can be lifted to a homomorphism $\bar{\rho} : \pi_1(U) \to G$.*

Proof. Set $Y_1 = M$ and $Y_2 = T \cup (F \times S^1)$, where T is a tree connecting the components of ∂M with a base point $q \in M$ inside M. Then $Y = Y_1 \cap Y_2 = T \cup (\sqcup_{i=1}^r S_i^1 \times S_i^1)$ is connected and $\pi_1(Y) \cong \Pi_{i=1}^r (\mathbb{Z}[l_i] \times \mathbb{Z}[m_i])$. Denote by γ_1 the edge of T, which connects the base point q with the first component of $\partial F \times S^1$. Let $g(F)$ be the genus of the surface F. Then $\pi_1(\gamma_1 \cup (F \times S^1))$ is presented by $2g(F) + r + 1$ generators $x_1, y_1, \ldots, x_g, y_g, l_1, \ldots, l_r, z$ and the relations $l_1 \cdots l_r = \Pi_{j=1}^{g(F)} [x_j, y_j]$, $zx_j = x_j z$, $zy_j = y_j z$, $zl_i = l_i z$, $i = 1, \ldots, r$, $j = 1, \ldots, g(F)$.

The elements l_i represent the longitudes, the letter $z = m_1$ denotes the meridian of the first component. To get $T \cup (F \times S^1)$ let us connect q with the ith component

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by an arc γ_i, $i = 1, \ldots, r$. This adds generators $u_1, \ldots, u_r$. Define the homomorphism $\varphi : \pi_1(T \cup (F \times S^1)) \to G$ by $\varphi(x_j) = \varphi(y_j) = \varphi(l_i) = e$, $\varphi(z) = \mu_1$, $\varphi(u_i) = \delta_{i_1}$, where $\delta_{i_1} \in G$ such that $\mu_i = \delta_{i_1}\mu_1\delta_{i_1}^{-1}$. The following diagram commutes

$$\begin{array}{ccc} \pi_1(Y) & \longrightarrow & \pi_1(Y_1) \\ \downarrow & & \downarrow \rho_M \\ \pi_1(Y_2) & \xrightarrow{\varphi} & G \end{array} \quad \text{since} \quad \begin{array}{ccc} l_i & \longrightarrow & l_i \\ \downarrow & & \downarrow \rho_M \\ l_i & \xrightarrow{\varphi} & e \end{array} \quad \text{and} \quad \begin{array}{ccc} m_i & \longrightarrow & m_i \\ \downarrow & & \downarrow \rho_M \\ u_i z u_i^{-1} & \xrightarrow{\varphi} & \mu_i \end{array}$$

□

Remark 4.15. It is in the proof of Lemma 4.14 that we need the hypothesis that the elements of the meridional system be conjugate to one another. Lemma 4.14 is essential for the well-definedness of the extended Johnson–Livingston product.

4.4. *Well-definedness of the extended Johnson–Livingston product*

Lemma 4.16. *Let* $(W, F, M\#_r N, \rho, f)$ *be a multi-connected sum of geometric pentads* $(U_M, F_M, M, \rho_M, f_M)$ *and* $(U_N, F_N, N, \rho_N, f_N)$ *corresponding to algebraic triples* $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$ *and* $(G, \boldsymbol{\mu}, \boldsymbol{\tau})$, *respectively. Assume that* $\boldsymbol{\lambda} \in (G')^r$ *and* $\boldsymbol{\tau} \in (G')^r$.

Using Lemma 4.13, *let* $\tilde{\rho}_M : \pi_1(U_M) \to G/G'$, $\tilde{\rho}_N : \pi_1(U_N) \to G/G'$ *and* $\tilde{\rho} : \pi_1(W) \to G/G'$ *be the homomorphisms extending* $\hat{\rho}_M, \hat{\rho}_N, \hat{\rho}$, *respectively.*

Let $\tilde{f}_M : U_M \to K(G/G', 1)$, $\tilde{f}_N : U_N \to K(G/G', 1)$ *and* $\tilde{f} : W \to K(G/G', 1)$ *be the corresponding maps. Then* $[W, \tilde{f}] = [U_M, \tilde{f}_M] + [U_N, \tilde{f}_N]$ *in* $\Omega_3(G/G')$.

Proof. Using the notations of Definition 4.5 and applying Lemma 4.12 for the group $H = G/G'$, we have

$$[W, \tilde{f}] - [U_M, \tilde{f}_M] = [(A \times [0,1]) \cup N \cup (I \times [0,1] \cup F_N) \times S^1, \tilde{f}_0],$$

where $\tilde{f}_0$ is the restriction of $\tilde{f}$ to $(A \times [0,1]) \cup N \cup (I \times [0,1] \cup F_N) \times S^1$. Applying Lemma 4.12 again, we get:

$$[W, \tilde{f}] - [U_M, \tilde{f}_M] - [U_N, \tilde{f}_N] = [(A \times [0,1]) \cup (I \times [0,1]) \times S^1, \tilde{f}_1],$$

where $\tilde{f}_1$ is the restriction of $\tilde{f}$ to $(A \times [0,1]) \cup (I \times [0,1]) \times S^1$. The latter manifold is a disjoint union of r copies of $S^2 \times S^1$ and any map $S^2 \times S^1 \to K(G,1)$ extends to $B^3 \times S^1$. Therefore, $[W, \tilde{f}] - [U_M, \tilde{f}_M] - [U_N, \tilde{f}_N] = 0$ in $\Omega_3(G/G')$. □

Theorem 4.17. *Let a group* G *have a meridional system* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in G^r$ *such that each* μ_i *is conjugate to* μ_1, $i = 2, \ldots, r$. *Assume that* $\lambda \in G^r$ *and that conditions* (i)–(iii) *of Theorems* 1.4, 1.5 *hold.*

(a) *The extended Johnson–Livingston product* $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\} \in Q(G)$ *is well-defined.*
(b) *We have* $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\} + \{\boldsymbol{\mu}, \boldsymbol{\tau}\} = \{\boldsymbol{\mu}, \boldsymbol{\lambda} \cdot \boldsymbol{\tau}\}$ *when defined.*

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Proof. (a) Let (U, F, M, ρ, f) and (U', F', M', ρ', f') be two geometric pentads representing the algebraic triple $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$. Then the pentad $(-U', -F', -M', \rho', f')$ represents $(G, \boldsymbol{\mu}, \boldsymbol{\lambda}^{-1})$. The multi-connected sum of (U, F, M, ρ, f) and $(-U', -F', -M', \rho', f')$ represents $(G, \boldsymbol{\mu}, \mathbf{e})$ by Lemma 4.8 and gives an element in $\mathrm{pr}_* H_3(G)$ by Lemma 4.14. Lemma 4.16 implies that $[U, \tilde{f}] - [U', \tilde{f}'] \in \mathrm{pr}_* H_3(G)$, hence $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\}$ is well-defined.

(b) The additivity is a direct consequence of Lemma 4.16. □

The following lemma will be used in Lemma 5.8, Sec. 5.3.

Lemma 4.18. *Suppose that conditions* (i), (ii) *of Theorem* 1.4 *hold for systems* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$ *and* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$. *Assume that* $\lambda_r = e$ *in the group* G. *Set* $\boldsymbol{\mu}' = (\mu_1, \ldots, \mu_{r-1})$ *and* $\boldsymbol{\lambda}' = (\lambda_1, \ldots, \lambda_{r-1})$. *Then the extended Johnson–Livingston product* $\{\boldsymbol{\mu}', \boldsymbol{\lambda}'\} \in Q(G)$ *is well-defined and equal to* $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\}$.

Proof. It is clear that $(G, \boldsymbol{\mu}', \boldsymbol{\lambda}')$ is an algebraic triple. Let (U', F', M', ρ', f') be a geometric pentad realizing it. Denote by F the surface obtained by removing the interior of a disk D^2 in the interior of F', hence $U' = (F \times S^1) \cup (D^2 \times S^1) \cup M'$.

Consider the disjoint union $M_0 = M' \sqcup (D^2 \times S^1)$. To make it connected, perform a 1-handle surgery. The resulting 3-dimensional manifold M satisfies $\pi_1(M) = \pi_1(M') * \mathbb{Z}$, so that ρ' can be extended to $\rho : \pi_1(M) \to G$ by sending the free generator to μ_r. Denote by $f : M \to K(G, 1)$ the corresponding map.

Set $U = M \cup (F \times S^1)$. It is easy to see that (U, F, M, ρ, f) is a geometric pentad corresponding to the triple $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$. One can then extend ρ' and ρ using Lemma 4.10 to $\tilde{\rho}' : \pi_1(U') \to G/G'$ and $\tilde{\rho} : \pi_1(U) \to G/G'$. Consider the associated maps $\tilde{f}' : U' \to K(G/G', 1)$ and $\tilde{f} : U \to K(G/G', 1)$.

Set $W = U \cup (F \times S^1 \times [0, 1]) \cup (-U')$, where $F \times S^1 \times \{0\}$ is identified with $F \times S^1 \subset U$ and $F \times S^1 \times \{1\}$ is identified with $F \times S^1 \subset (-U')$. The compact 4-manifold W has boundary $\partial W = U \cup (-U')$. Moreover, the maps $\tilde{f}$ and $\tilde{f}'$ extend to W since they agree on $F \times S^1$. Hence, the equality $[U', \tilde{f}'] = [U, \tilde{f}]$ holds in $\Omega_3(G/G')$. This shows that $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\} = \{\boldsymbol{\mu}', \boldsymbol{\lambda}'\}$ in $Q(G)$. □

5. Realizable Systems: Proof of Theorem 1.5

Let $\boldsymbol{\mu}$ be a meridional system of G. In this section, we still assume that the elements μ_i of the system $\boldsymbol{\mu}$ are conjugate to one another in G.

5.1. *Necessity in Theorem 1.5*

Proposition 2.3 implies that the set $R(G, \boldsymbol{\mu})$ of realizable systems is non-empty.

Lemma 5.1. *If* $\boldsymbol{\lambda} \in R(G, \boldsymbol{\mu})$, *then condition* (iii) *of Theorem* 1.5 *holds.*

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Proof. Let $L = \cup_{i=1}^r L_i \subset S^3$ be a link realizing the system $\boldsymbol{\lambda} \in G^r$. In other words, one has $\rho(m_i) = \mu_i$ and $\rho(l_i) = \lambda_i$, where $m_1, \ldots, m_r$ are meridians of $L_1, \ldots, L_r$, $(l_1, \ldots, l_r)$ is a preferred system of longitudes for the link L. By Lemma 2.2, in the group $H_1(S^3 - L)$ one has $[l_i] = \sum_{j=1}^r \alpha_{ij}[m_j]$, where $\alpha_{ij} = \mathrm{lk}(L_i, L_j)$, $i \neq j$ and $\alpha_{ii} = -\sum_{j\neq i} \mathrm{lk}(L_i, L_j)$.

The homomorphism $\hat{\rho} = \mathrm{pr} \circ \rho : \pi_1(S^3 - L) \to G \to G/G'$ factorizes through $H_1(S^3 - L)$. Hence in the quotient G/G' one gets $[\lambda_i] = \sum_{j=1}^r \alpha_{ij}[\mu_j]$. By the conditions of Theorem 1.5 one has $[\mu_1] = \cdots = [\mu_r]$. Since $\sum_{j=1}^r \alpha_{ij} = 0$, then $[\lambda_i] = 0$ for each $i = 1, \ldots, r$. □

Lemma 5.2. *If $\boldsymbol{\lambda} \in R(G, \boldsymbol{\mu})$, then condition* (iv) *of Theorem* 1.5 *holds.*

Proof. By Lemma 5.1 and Theorem 4.17(a) the Johnson–Livingston product $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\}$ is well-defined. Let $L = \cup_{i=1}^r L_i \subset S^3$ be a link with a surjective homomorphism $\rho : \pi_1(S^3 - L) \to G$ realizing the given system $\boldsymbol{\lambda} \in G^r$. Denote by $f : S^3 - L \to K(G, 1)$ a continuous map corresponding to ρ.

Let $F \subset S^3$ be an oriented surface such that $F \cap L = \partial F = L$. Consider the sphere S^3 as the boundary of the ball B^4. Push F into B^4 leaving ∂F in S^3. Take a sufficiently small regular neighborhood $T(F) \subset B^4$. Then the complement $V = B^4 - \mathrm{Int}\, T(F)$ is an oriented compact 4-dimensional manifold with boundary $\partial V = (S^3 - \cup_{i=1}^r \mathrm{Int}\, T(L_i)) \cup (S^1 \times F)$.

The pentad $(\partial V, F - \mathrm{Int}\, T(L), S^3 - \mathrm{Int}\, T(L), \rho, f)$ is a geometric pentad corresponding to $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$. Note that $H_1(V) \cong \mathbb{Z}$. Denote by j the inclusion $j : S^3 - \mathrm{Int}\, T(L) \to V$. Consider the homomorphism $\tilde{\rho} : \pi_1(V) \to G/G'$ induced by the map which sends the generator of $H_1(V)$ to $[\mu_1]$. This makes the following diagram commutative

$$\begin{array}{ccc} \pi_1(S^3 - \mathrm{Int}\, T(L)) & \xrightarrow{\mathrm{pr}\circ\rho} & G/G' \\ {\scriptstyle j_*}\downarrow & & \| \\ \pi_1(V) & \xrightarrow{\tilde{\rho}} & G/G' \end{array}$$

Denote by i_* the homomorphism induced by the inclusion $\partial V \subset V$. Let $\tilde{f} : \partial V \to K(G/G', 1)$ be the map corresponding to $\tilde{\rho} \circ i_*$. The construction above shows that $[\partial V, \tilde{f}]$ vanishes in the group $\Omega_3(G/G')$. This implies $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\} = 0$. □

5.2. *Partial realization results*

Here we shall realize some auxilary systems needed for sufficiency in Theorem 1.5. For any meridional system $\boldsymbol{\mu} \in G^r$, we have a well-defined homomorphism $\theta_{\boldsymbol{\mu}} : Z(\boldsymbol{\mu}) \to H_2(G)$, $\theta_{\boldsymbol{\mu}}(\boldsymbol{\lambda}) = \sum_{i=1}^r \langle \mu_i, \lambda_i \rangle$. Here $Z(\boldsymbol{\mu}) \subset G^r$ is the centralizer subgroup of $\boldsymbol{\mu} \in G^r$. Denote by $P(G, \boldsymbol{\mu}) \subset Z(\boldsymbol{\mu})$ the kernel of $\theta_{\boldsymbol{\mu}}$. Therefore, a system $\boldsymbol{\lambda} \in G^r$ is weakly realizable if and only if $\boldsymbol{\lambda} \in P(G, \boldsymbol{\mu})$.

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Since the elements μ_i of the system $\boldsymbol{\mu}$ are conjugate to one another in G, the abelianization G/G' is a cyclic group. We shall denote by n its order. In particular, $n = 0$ when $G/G' \cong \mathbb{Z}$ and $n = 1$ when $G = G'$. The following lemma is a generalization of [6, Theorem 3].

Lemma 5.3. *Let $\boldsymbol{\mu} \in G^r$ be a meridional system of a finitely generated group G. If condition* (iii) *of Theorem* 1.5 *holds and $\boldsymbol{\lambda} \in P(G, \boldsymbol{\mu})$, then there are $a_1, \ldots, a_r \in \mathbb{Z}$ such that $\lambda_{\mathbf{a}} := (\lambda_1 \mu_1^{a_1}, \ldots, \lambda_r \mu_r^{a_r}) \in R(G, \boldsymbol{\mu})$ and $a_i \equiv 0 \pmod{n}$.*

Proof. By Lemma 3.9(b), there is a link $L' \subset S^3$ with a surjective homomorphism $\rho' : \pi_1(S^3 - L') \to G$ realizing a system $\boldsymbol{\lambda}_{\mathbf{a}} := (\lambda_1 \mu_1^{a_1}, \ldots, \lambda_r \mu_r^{a_r})$ with $a_1, \ldots, a_r \in \mathbb{Z}$. For $i = 1, \ldots, r$, condition (iii) of Theorem 1.5 says that $[\lambda_i] = 0$ in G/G' and Lemma 5.1 shows that $[\lambda_i] + a_i[\mu_i] = 0$ in G/G'. Since $[\mu_1]$ generates G/G', we get $a_i \equiv 0 \pmod{n}$. □

Lemma 5.4. *Let $\boldsymbol{\mu} \in G^2$ be a meridional system of a finitely generated group G. Let $L = L_1 \cup L_2 \subset S^3$ be a 2-component link with a surjective homomorphism $\rho : \pi_1(S^3 - L) \to G$ such that $\rho(m_1) = \mu_1$, $\rho(m_2) = \mu_2$, where m_1, m_2 are the meridians of L_1, L_2.*

Let b, c be integers such that $b + c \equiv 0 \pmod{n}$. Then there is an oriented trivial knot $J \subset S^3 - L$ such that $\rho(J) = e$, $\mathrm{lk}(J, L_1) = b$, $\mathrm{lk}(J, L_2) = c$.

Proof. Let us produce two oriented curves $J_1, J_2 \subset S^3 - L$ such that

- $\rho(J_1) = \mu_1^b \mu_2^c$, $\mathrm{lk}(J_1, L_1) = b$, $\mathrm{lk}(J_1, L_2) = c$;
- $\rho(J_2) = \mu_1^b \mu_2^c$, $\mathrm{lk}(J_2, L_1) = 0$, $\mathrm{lk}(J_2, L_2) = 0$.

To construct J_1 choose a closed curve in $S^3 - L$ homotopic to $m_1^b m_2^c$, perform a homotopy to make it embedded. Since $\mu_1^b \mu_2^c \in G'$ and G is normally generated by μ_1, one gets $\mu_1^b \mu_2^c = \prod_{i=1}^s \xi_i \mu_1^{\varepsilon_i} \xi_i^{-1}$, where the ξ_i are in G and the ε_i are integers such that $\sum_{i=1}^s \varepsilon_i = 0$ (see [6, Claim in the proof of Theorem 2]).

Let γ_i, $i = 1, \ldots, s$ be closed curves in $S^3 - L$ such that $\rho(\gamma_i) = \xi_i$. The curve $\gamma = \prod_{i=1}^s \gamma_i \mu_1 \gamma_i^{-1}$ can be homotoped to a simple closed curve J_2 in $S^3 - (L \cup J_1)$ such that $\rho(J_2) = \mu_1^b \mu_2^c$ and $\mathrm{lk}(J_2, L_1) = \mathrm{lk}(J_2, L_2) = 0$. Consider the connected sum of J_1 and J_2 in $S^3 - L$. It can be unknotted by a homotopy in $S^3 - L$. The resulting curve J satisfies the required properties. □

The following lemma is a generalization of [6, Theorem 2] to 2-component links.

Lemma 5.5. *Let $\boldsymbol{\mu} \in G^2$ be a meridional system of a finitely generated group G. For all integers b, c such that $b + c \equiv 0 \pmod{n}$, one has $(\mu_1^{b^2+bc}, \mu_2^{bc+c^2}) \in R(G, \boldsymbol{\mu})$.*

Proof. By Proposition 4.11(a) there is a 2-component algebraically split link $L = L_1 \cup L_2$ realizing the system $(e, e) \in G^2$. Let m_1, m_2 be the meridians of

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L_1, L_2 and denote by $\bar{l}_1, \bar{l}_2$ the preferred longitudes of L_1, L_2, respectively. Since L is algebraically split, $(\bar{l}_1, \bar{l}_2)$ is the preferred system of longitudes for L.

Take a homomorphism $\rho : \pi_1(S^3 - L) \to G$ such that $\rho(m_1) = \mu_1$, $\rho(m_2) = \mu_2$, $\rho(\bar{l}_1) = \rho(\bar{l}_2) = e$. By Lemma 5.4 there exists an oriented trivial knot $J \subset S^3 - L$ such that $\rho(J) = e$, $\mathrm{lk}(J, L_1) = b$, $\mathrm{lk}(J, L_2) = c$.

The integral surgery on J with framing $+1$ carries S^3 to itself and the link L to a link $L' = L'_1 \cup L'_2$ with a surjective homomorphism $\rho' : \pi_1(S^3 - L') \to G$. The meridians of L' satisfy $m'_1 = m_1$, $m'_2 = m_2$. Let us denote by $\overline{(l'_1, l'_2)}$ the preferred system of longitudes of L'. Our aim is to prove that $\rho'(l'_1) = \mu_1^{b^2+bc}$ and $\rho'(l'_1) = \mu_2^{bc+c^2}$.

Let m_J be the meridian of J and let l_J be a longitude such that $\mathrm{lk}(l_J, J) = +1$. The homology class of l_J in the group $H_1(S^3 - L - J) \cong \mathbb{Z}[m_1] \oplus \mathbb{Z}[m_2] \oplus \mathbb{Z}[m_J]$ is given in this basis by the vector $[l_J] = \begin{pmatrix} b \\ c \\ 1 \end{pmatrix}$. Similarly, $H_1(S^3 - L') \cong \mathbb{Z}[\mu'_1] \oplus \mathbb{Z}[\mu'_2]$.

In these bases the attaching map of the surgery disc is given by $\begin{pmatrix} b \\ c \\ 1 \end{pmatrix}$.

The inclusion $\psi : S^3 - (L \cup J) \to S^3 - L'$ is described in homology by $\begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & -c \end{pmatrix}$.

Denote by $\bar{l}'_1, \bar{l}'_2$ the preferred longitudes of the components L'_1, L'_2, respectively. One gets $[\bar{l}_1] = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$, $[\bar{l}_2] = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \in H_1(S^3 - L - J)$;

$$\psi_*([\bar{l}_1]) = \begin{pmatrix} -b^2 \\ -bc \end{pmatrix}, \; \psi_*([\bar{l}_2]) = \begin{pmatrix} -bc \\ -c^2 \end{pmatrix}, \quad [\bar{l}'_1] = \begin{pmatrix} 0 \\ * \end{pmatrix}, \quad [\bar{l}'_2] = \begin{pmatrix} * \\ 0 \end{pmatrix} \in H_1(S^3 - L').$$

In particular, for the link L', one gets $\mathrm{lk}(L'_1, L'_2) = -bc$ and $l'_1 = m_1^{bc}\bar{l}'_1$, $l'_2 = m_2^{bc}\bar{l}'_2$. Since $\psi(\bar{l}_1)$ and $\psi(\bar{l}_2)$ are longitudes of the components L'_1 and L'_2, then $\bar{l}'_1 = m_1^{d_1}\psi(\bar{l}_1)$ and $\bar{l}'_2 = m_2^{d_2}\psi(\bar{l}_2)$ for $d_1, d_2 \in \mathbb{Z}$. By substituting in the vectors above one obtains $d_1 = b^2$, $d_2 = c^2$. One computes $l'_1 = m_1^{b^2+bc}\psi(l_1)$ and $l'_2 = m_2^{bc+c^2}\psi(l_2)$. Since $\rho(l_1) = \rho(l_2) = e$, we get $\rho'(l'_1) = \mu_1^{b^2+bc}$ and $\rho'(l'_1) = \mu_2^{bc+c^2}$ as desired. □

Lemma 5.6. *Let $\boldsymbol{\mu} \in G^2$ be a meridional system of a finitely generated group G.*

(a) $(1, \mu_2^{n^2}) \in R(G, (\mu_1, \mu_2))$, *where n is the order of G/G'.*
(b) *For any integer $h \in \mathbb{Z}$, $(\mu_1^{hn}, \mu_2^{-hn}) \in R(G, \boldsymbol{\mu})$.*

Proof. (a) [6, Theorem 2] states that, in the particular case $r = 1$ of a knot, μ^{n^2} is realizable, i.e. there is a knot K with a surjective homomorphism $\rho : \pi_1(S^3 - K) \to G$ such that $\rho(m) = \mu$, $\rho(l) = \mu^{n^2}$, where m and l are the meridian and the preferred longitude of K. By Proposition 4.11(b) the link $O \sqcup K$ realizes $(1, \mu_2^{n^2})$, where O is the trivial circle.

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(b) Lemma 5.5 with $b = h$, $c = n - h$ implies $(\mu_1^{hn}, \mu_2^{n^2-hn}) \in R(G, \boldsymbol{\mu})$. By Proposition 4.11(a) the set $R(G, \boldsymbol{\mu})$ is a group. Then we get

$$(\mu_1^{hn}, \mu_2^{-hn}) = (\mu_1^{hn}, \mu_2^{n^2-hn}) \cdot (1, \mu_2^{n^2})^{-1} \in R(G, \boldsymbol{\mu}). \qquad \square$$

5.3. *Sufficiency in Theorem 1.5*

Here we finish the proof of our main Theorem 1.5 formulated in Sec. 1.3.

Proposition 5.7. *Under the conditions of Lemma* 5.3 *there exists* $h \in \mathbb{Z}$ *such that* $(\lambda_1\mu_1^{hn}, \lambda_2, \ldots, \lambda_r) \in R(G, \boldsymbol{\mu})$.

Proof. By Lemma 5.3 take integers $a_1, \ldots, a_r \in \mathbb{Z}$ such that $\lambda_{\mathbf{a}} = (\lambda_1\mu_1^{a_1}, \ldots, \lambda_r\mu_r^{a_r}) \in R(G, \boldsymbol{\mu})$ and $a_i \equiv 0 \pmod n$ for each $i = 1, \ldots, r$. If $n = 0$, then the result is obvious. Suppose $n \geq 1$ and write $a_i = h_i n$, $h_i \in \mathbb{Z}$, $i = 1, \ldots, r$.

By Lemma 5.6(b) the system $(\mu_1^{h_2n}, \mu_1^{-h_2n})$ belongs to $R(G, (\mu_1, \mu_2))$, hence $(\mu_1^{h_2n}, \mu_2^{-h_2n}, 1, \ldots, 1) \in R(G, \boldsymbol{\mu})$ by Proposition 4.11(b). Then we obtain

$$\boldsymbol{\lambda}'_{\mathbf{a}} := (\lambda_1\mu_1^{(h_1+h_2)n}, \lambda_2, \lambda_3\mu_3^{h_3n}, \ldots, \lambda_r\mu_r^{h_rn}) = \boldsymbol{\lambda}_{\mathbf{a}} \cdot (\mu_1^{h_2n}, \mu_2^{-h_2n}, 1, \ldots, 1) \in R(G, \boldsymbol{\mu}).$$

Apply the same trick to the components $\lambda_1\mu_1^{(h_1+h_2)n}$ and $\lambda_3\mu_3^{h_3n}$ of $\boldsymbol{\lambda}'_{\mathbf{a}}$ to kill h_3 and so on. Finally, we get $h \in \mathbb{Z}$ such that $(\lambda_1\mu_1^{hn}, \lambda_2, \ldots, \lambda_r) \in R(G, \boldsymbol{\mu})$ as required. $\square$

The following lemma is a simple generalization of [6, Theorem 4].

Lemma 5.8. *Let* $\boldsymbol{\mu} \in G^r$ *be a meridional system of a finitely generated group* G. *Let* h *be an integer such that* $\{\boldsymbol{\mu}, \boldsymbol{\mu}_h\} = 0$, *where* $\boldsymbol{\mu}_h = (\mu_1^{hn}, 1, \ldots, 1)$. *Then* $\boldsymbol{\mu}_h \in R(G, \boldsymbol{\mu})$.

Proof. By Lemma 4.18 one has $\{\boldsymbol{\mu}, \boldsymbol{\mu}_h\} = \{\mu_1, \mu_1^{hn}\} \in Q(G)$. [6, Theorem 4] says that $\mu_1^{hn} \in R(G, \mu_1)$. Then Proposition 4.11(b) implies $\boldsymbol{\mu}_h \in R(G, \boldsymbol{\mu})$. $\square$

Proof of Theorem 1.5. Theorem 1.5 says that, for a system $\boldsymbol{\lambda} \in P(G, \boldsymbol{\mu})$, conditions (iii) and (iv) are equivalent to realizability $\boldsymbol{\lambda} \in R(G, \boldsymbol{\mu})$. Necessity of these conditions follows from necessity in Theorem 1.4 and Lemmas 5.1, 5.2.

To prove sufficiency take by Proposition 5.7 an integer $h \in \mathbb{Z}$ such that $\boldsymbol{\lambda}_h \in R(G, \boldsymbol{\mu})$, where $\boldsymbol{\lambda}_h = (\lambda_1\mu_1^{hn}, \lambda_2, \ldots, \lambda_r)$. By Lemma 5.2 (necessity of condition (iv)) one gets $\{\boldsymbol{\mu}, \boldsymbol{\lambda}_h\} = 0$ in $Q(G)$. By Theorem 4.17(b) (additivity of the extended Johnson–Livingston product) one has $0 = \{\boldsymbol{\mu}, \boldsymbol{\lambda}_h\} = \{\boldsymbol{\mu}, \boldsymbol{\lambda}\} + \{\boldsymbol{\mu}, \boldsymbol{\mu}_h\}$, where $\boldsymbol{\mu}_h = (\mu_1^{hn}, 1, \ldots, 1)$.

Condition (iv) of Theorem 1.5 means that $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\} = 0$, hence one obtains $\{\boldsymbol{\mu}, \boldsymbol{\mu}_h\} = 0$. Then by Lemma 5.8 we get $\boldsymbol{\mu}_h \in R(G, \mu)$. Since by Proposition 4.11(a) the set $R(G, \boldsymbol{\mu})$ is a group, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_h \cdot \boldsymbol{\mu}_h^{-1} \in R(G, \boldsymbol{\mu})$ as required. $\square$

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6. Applications

We give below examples of groups where conditions (i)–(iv) of Theorems 1.4 and 1.5 apply.

Example 6.1. Let G be the group of a classical knot K and let $\mu_1, \ldots, \mu_r$ be meridians of K. Then $\boldsymbol{\mu} = (\mu, \ldots, \mu_r) \in G^r$ is a meridional system of G and the only nontrivial conditions of Theorems 1.4 and 1.5 are conditions (i) and (iii) since it is well-known that $H_2(G) = 0$ and $H_3(G/G') = H_3(\mathbb{Z}) = 0$. For $i = 1, \ldots, r$, let λ denote the preferred longitude of K that commutes with μ_i. Then all systems of the form $\boldsymbol{\lambda} = (\lambda^{a_1}, \ldots, \lambda^{a_r})$, $a_i \in \mathbb{Z}$, $i = 1, \ldots, r$ are realizable.

If K is a hyperbolic knot, then one shows that these are the only realizable systems since any element of G which commutes with μ_i is parabolic and belongs to the peripheral subgroup of G containing μ_i.

If K is a composite knot, then there are other realizable systems because the preferred longitude of a summand of K commutes with the meridian of K but is not a power of the preferred longitude of K.

Example 6.2. Consider virtual knot groups as described by Kim [8]. Let G be the group of a virtual knot such that $H_2(G)$ is cyclic of order 2. Such a knot exists [8, Sec. 6.3]. Its group has the presentation:

$$\langle a, b \mid b = a^{-1}b^2ab^{-2}a,\ b = [ba^{-1}, a^{-1}b]^{-1}b[ba^{-1}, a^{-1}b]\rangle.$$

Let $r > 0$ be an integer and let μ and λ denote the meridian and preferred longitude of the virtual knot. Set $\boldsymbol{\mu} = (\mu, \ldots, \mu)$ and $\boldsymbol{\lambda} = (\lambda, \ldots, \lambda)$ in G^r. Then $\boldsymbol{\mu}$ is a meridional system of G, conditions (i), (iii) and (iv) are satisfied, in particular, $G/G' \cong \mathbb{Z}$ so that $H_3(G/G') = 0$. The Pontryagin product $\langle \mu, \lambda \rangle$ generates $H_2(G)$ by [8, Theorem 15], hence condition (ii) is equivalent to the condition that r is even.

Example 6.3. Let G be the group described in [6, Appendix 3]. It has the presentation

$$\langle a, b, x, y \mid b^{-1}a^nb = [x, y],\ b^{-2}a^nb^2 = y^{-2}xy^2,\ a^{-1}ba = y^{-3}xy^3,\ a^{-2}ba^2 = x^{-2}yx^2\rangle,$$

where n is a positive integer. The group G is normally generated by a and G/G' is cyclic of order n generated by the class of a. According to [6], we have $H_2(G) = H_3(G) = 0$ and since $G/G' \cong \mathbb{Z}_n$, we get $Q(G) = H_3(G/G') \cong \mathbb{Z}_n$.

Let r be a positive integer and take $\boldsymbol{\mu} = (a, \ldots, a) \in G^r$ and $\boldsymbol{\lambda} = (a^n, \ldots, a^n) \in (G')^r$. Conditions (i)–(iii) are satisfied. We shall compute explicitly the extended Johnson–Livingston product $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\}$ and show that it is r times the generator of $Q(G) \cong \mathbb{Z}_n$. So, condition (iv) is equivalent to n divides r.

Let $L^0 \subset S^3$ be the trivial link with r components. Denote by m_i^0 and l_i^0, $i = 1, \ldots, r$ the meridians and preferred longitudes of L^0. Let $\rho_0 : \pi_1(S^3 - L^0) \to G$ be the homomorphism sending m_i^0 to a. Let M be the connected sum of $S^3 - \operatorname{Int} T(L^0)$

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with three copies of $S^2 \times S^1$. We can extend ρ_0 to a surjective homomorphism $\rho : \pi_1(M) \to G$. Let $f : M \to K(G, 1)$ be a corresponding continuous map.

Parametrize the components of $\partial T(L^0)$ by maps $g_i : S^1 \times S^1 \to \partial T(L^0)$ in such a way that $g_i(\{\text{pt}\} \times S^1) = m_i^0$ and $g_i(S^1 \times \{\text{pt}\}) = (m_i^0)^n l_i^0$, $i = 1, \ldots, r$. The geometric triple (M, ρ, f) realizes the algebraic triple $(G, \boldsymbol{\mu}, \boldsymbol{\lambda})$. Let F denote the 2-dimensional sphere punctured with r holes. Glue $F \times S^1$ to M to get a closed manifold U as described in Definition 4.3 and extend ρ to $\rho_M : \pi_1(U) \to G/G'$ with associated map $f : U \to K(G/G', 1)$.

The way that ρ_M is constructed in Lemma 4.14 shows that we may assume that the restriction of $\tilde{f}$ to $F \times S^1$ is of the form $\tilde{f}(x, t) = g(t)$ for $x \in F$ and $t \in S^1$, where $g : S^1 \to K(G/G', 1)$ is a map such that in homotopy the generator of $\pi_1(S^1)$ is sent to the class of a in G/G'. The associated pentad computes the extended Johnson–Livingston product $\{\boldsymbol{\mu}, \boldsymbol{\lambda}\}$.

Set $U_1 = F \times S^1$ and $U_2 = (S^2 - \operatorname{Int} F) \times S^1$ which is a disjoint union of r solid tori. Lemma 4.12 shows that, in $\Omega_3(G/G')$, the element $[U, \tilde{f}]$ is equal to $[U', \tilde{f}']$, where U' is a connected sum of lens spaces of type $L(n, 1)$ and $\tilde{f}'$ is the extension of f to U'. This uses the fact that any element of the form $[S^2 \times S^1, h]$ vanishes in $\Omega_3(G/G')$. Therefore, $[U, \tilde{f}] = r[L(n, 1), \psi]$ in $\Omega_3(G/G')$, where $\psi : L(n, 1) \to K(G/G', 1)$ induces a surjective homomorphism $\pi_1(L(n, 1)) \cong \mathbb{Z}_n \to G/G'$.

The result follows since the element $[L(n, 1), \psi]$ corresponds to a generator of $H_3(G)$, see [6, Sec. 3, p. 138].

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