Configuration Spaces

of round robots moving on metric graphs

Vitaliy Kurlin, http://kurlin.org
Microsoft Research Cambridge
and Durham University, UK

Joint work in progress

with former student Dr Marjan Safi-Samghabadi

PhD *Collision-free motions of round robots on metric graphs*, Durham University, June 2013.

Short version in Proceedings of ICRoM 2014:

Int. Conference on Robotics and Mechatronics.

Motivations: why graph?

Real robots often have restricted movements:

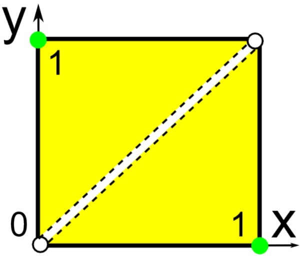
- trains on fixed railway paths
- factory robots on floor tracks
- biological cells in blood vessels
- elementary particles in accelerators

A theoretical model of robots

n robots are points $x_1, \ldots, x_n$ in a graph G.

Def: a configuration $(x_1, \ldots, x_n)$ is safe if all positions are different: $x_i \neq x_j$ for any $i \neq j$.

Example: $G = [0, 1]$ interval, $n = 2$ robots may have the safe configurations $(x_1, x_2) = (0, 1)$ or $(1, 0)$.



Configuration space $\mathrm{OC}_n(G)$

Topology enters the scene when we study a space of all objects, not a single object.

Def: the ordered configuration space of all safe configurations of n *ordered* points in a graph G is $\mathrm{OC}_n(G) = \{(x_1, \ldots, x_n) \in G^n \mid x_i \neq x_j\}$.

Example: $\mathrm{OC}_2([0,1])$ is the square $[0,1]^2$ without the diagonal $\{x = y\} \subset [0,1]^2$.

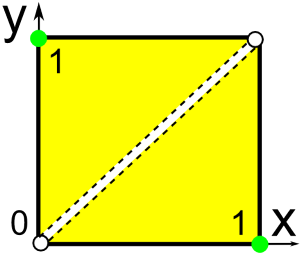
In general, the space $\mathrm{OC}_n(G)$ is n-dimensional.

Unordered space $\mathrm{UC}_n(G)$

Def: the unordered configuration space of all safe configurations of *n indistinguishable* points

is the quotient of the ordered space $\mathrm{OC}_n(G)$ by the action of the permutation group S_n.

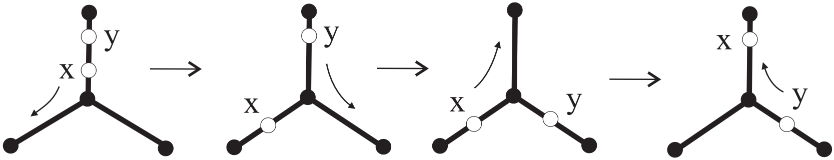
Example: $\mathrm{OC}_2([0,1])$ consists of 2 triangles $\{x < y\}$ and $\{x > y\}$, $\mathrm{UC}_2([0,1])$ is 1 triangle.



Connectivity of $\mathrm{OC}_n(G)$ and $\mathrm{UC}_n(G)$

Problem: understand the topology of $\mathrm{OC}_n(G)$ up to continuous deformations (homotopy).

Lemma: if G is connected and has a vertex of deg ≥ 3, then $\mathrm{OC}_n(G), \mathrm{UC}_n(G)$ are connected.



We can swap robots near a vertex of deg ≥ 3.

Simplify configuration spaces

$\mathrm{OC}_n(G)$ is non-compact, but continuously deforms to a small compact subspace.

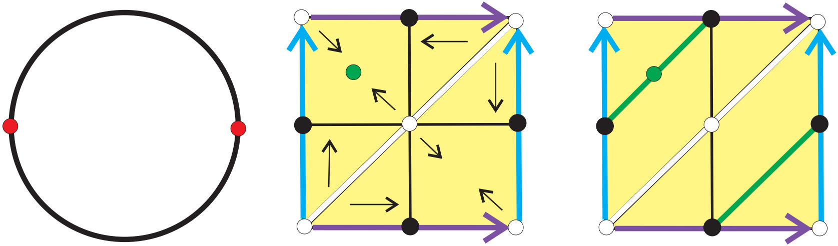
Example: $\mathrm{OC}_2([0,1])$ of 2 triangles continuously deforms to (has the homotopy type of) 2 points.

$\mathrm{UC}_n([0,1])$ is contractible to $\{\frac{i}{n}\}$, $i = 1, \ldots, n$.

$\mathrm{OC}_n([0,1])$ has $n!$ contractible components.

From a torus to a circle

$\mathrm{OC}_2(S^1)$ is a torus without a diagonal. A torus is obtained from $[0,1]^2$ by gluing opposite sides.



We can continuously deform $\mathrm{OC}_2(S^1)$ to S^1.

The space of 2 robots in S^1 looks like a circle.

A small compact subcomplex

Th (R. Ghrist): if $G \not\approx S^1$ has k vertices of deg ≥ 3, the configuration space $\mathrm{OC}_n(G)$ continuously deforms to a *k-dim complex*.

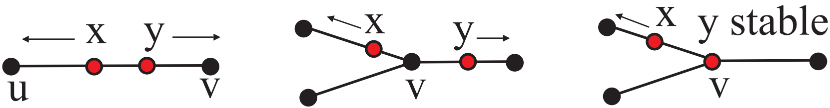
The dimension k of the simplified space depends only on the graph G, not on n.

Example: for a k-star T_k, $OC_2(T_k) \sim$ a wedge of $k^2 - 3k + 1$ circles. $\mathrm{OC}_n(T_k) \sim$ a wedge of $1 + (kn - 2n - k + 1)(n + k - 2)!/(k - 1)!$ circles.

Idea: find the Euler characteristic of $\mathrm{OC}_n(T_k)$.

Repelling robots can't be close

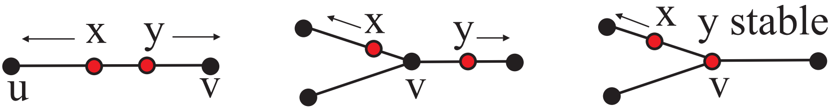
Real-life robots should be placed at a *distance away* from each other. We call robots close if they are in the same edge or in adjacent edges.



To control distances between robots, *subdivide* long stretches of roads into shorter edges.

Discrete configuration spaces

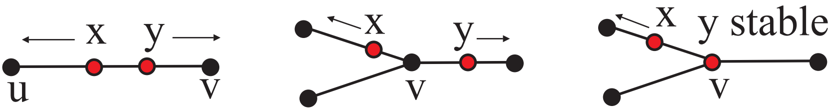
Def: the discrete space $\mathrm{OD}_n(G)$ consists of $(x_1, \ldots, x_n)$ with $\mathrm{supp}(x_i) \cap \mathrm{supp}(x_j) = \emptyset$, where the support $\mathrm{supp}(x) = x$ if x is a vertex, else $\mathrm{supp}(x)$ is the (closed) edge $\bar{e}$ containing x.



Almost always $\mathrm{OC}_n(G)$ continuously deforms to the smaller compact subspace $\mathrm{OD}_n(G)$.

Making a space compact

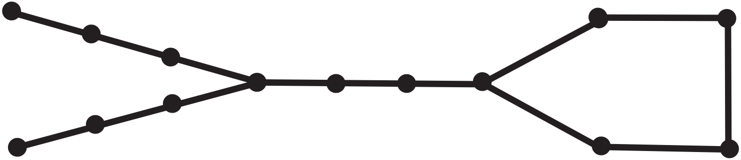
Case $n = 2$. If G has no loops and multiple edges, then $\mathrm{OC}_2(G)$ deforms to $\mathrm{OD}_2(G)$ and keeps all essential homotopy properties.



We move 2 robots away from each other until $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$. If robots are far away from each other, keep them at their positions.

Sufficient subdivisions

Th (Abrams): $\mathrm{OC}_n(G)$ *continuously deforms* to $\mathrm{OD}_n(G)$ if any path between *non-trivial* vertices (of degree $\neq 2$) has at least $\geq n+1$ edges.

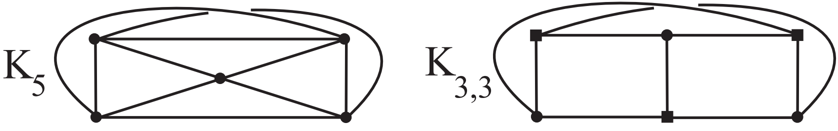


Here is a *necessary and sufficient* condition:

$\geq n-1$ edges between vertices of deg $\neq 2$ and $\geq n+1$ edges in any cycle ($n=4$ in the picture).

Kuratowski graphs K_5 and $K_{3,3}$

$K_5, K_{3,3}$ are the only graphs G such that $\mathrm{OC}_2(G)$ continuously deforms to a compact surface.

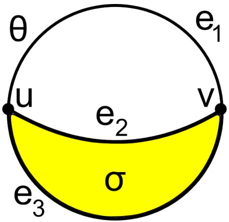


$\mathrm{OD}(K_5, 2) = \cup$ 10 triangular tubes $\approx$ the orientable surface of genus 6 without boundary.

$\mathrm{OD}(K_{3,3}, 2) = \cup$ 9 square tubes $\approx$ the orientable surface of genus 4 without boundary.

Homology group of a complex

Def: a complex C consists of edges, triangles etc. The homology group $H_1(C)$ is the quotient of 1-cycles of C over boundaries of 2-cells in C.



$H_0(\theta) = \mathbb{Z}_2$. $H_1(\theta) = \mathbb{Z}_2$ as $\{\text{cycles } e_1 + e_2, e_2 + e_3, e_1 + e_3\}$ over $\{\partial\sigma = e_2 + e_3\}$. $H_2(\theta) = 0$.

$\dim H_k = \#$ linearly independent cycles of C.

Abelian group $H_1(C)$ is a topological invariant.

More topology of $\mathrm{OC}_n(G), \mathrm{UC}_n(G)$

K.H.Ko, H.W.Park, 2012: the homology group H_1 of $\mathrm{UC}_n(G)$ and $\mathrm{OC}_2(G)$ is *computed*

in terms of graph-theoretic invariants of G, e.g. rank of H_1, $\#$ bi-connected components etc.

The *fundamental group* π_1 of loops (up to homotopy) is a non-abelian version of H_1 and the *braid group* of n strings on a graph G.

VK, 2012: algorithm for writing an explicit presentation of the braid group $\pi_1(\mathrm{UC}_n(G))$.

A more practical model

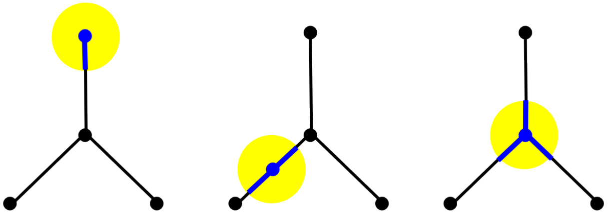
Real robots are not points, but have *real sizes* and move along tracks of some finite length.

Def: a metric graph G has edges of lengths $a_{ij} > 0$. The distance $d(x, y)$ between any points $x, y \in G$ is the length of a shortest path $x \to y$.

For simplicity, let G have no multiple edges.

A round robot in a metric graph

Def: a round robot in a metric graph G is a metric ball $B(a; r) = \{x \in G \ : \ d(a, x) \leq r\}$.



G is an abstract graph, not embedded in $\mathbb{R}^N$, so $B(a; r) \subset G$ is a union of arcs, not a round ball.

Space of 2 ordered round robots

Def: the configuration space of 2 ordered robots of a radius $r > 0$ in a given metric graph G is $\mathrm{OC}(G; r) = \{(x, y) \in G \times G \ : \ d(x, y) \geq 2r\}$.

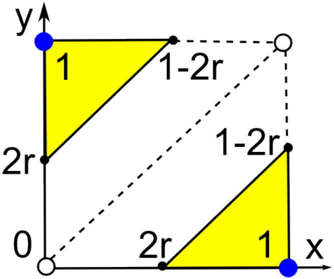
$d(x, y)$ is the distance between centres, so robots may touch each other. A *collision-free* motion of 2 robots is a path in $\mathrm{OC}(G; r)$.

More general: $\mathrm{OC}_n(G; r_1, \ldots, r_n)$ for n robots with n radii, unordered space $\mathrm{UC}_n(G; r_1, \ldots, r_n)$.

2 ordered round robots on $[0,1]$

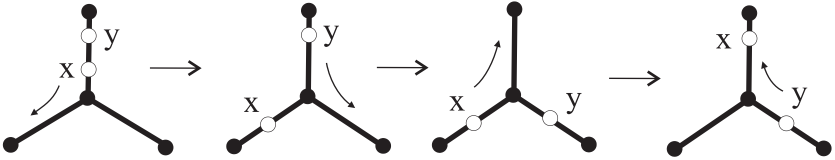
When the radius r is increasing from 0 to $\frac{1}{2}$, the configuration space $\mathrm{OC}([0,1];r)$ is shrinking from $[0,1]^2 - \{x = y\}$ to $\{(0,1),(1,0)\}$.

The homotopy types of $\mathrm{OC}([0,1],r)$: 2 contractible components for $0 \le r \le \frac{1}{2}$, empty for any radius $r > \frac{1}{2}$.



2 round robots on a tripod T

Let 3 edges of the tripod T have length 1.



The homotopy types of $\mathrm{OC}(T, r)$ for different r:

- continuously deforms to S^1 for $0 \leq r \leq \frac{1}{2}$,
- 6 contractible components for $\frac{1}{2} \leq r \leq 1$,
- $\mathrm{OC}(T, r)$ is empty for any radius $r > 1$.

Critical radii of Kenneth Deeley

Kenneth Deeley's PhD supervised by Prof Michael Farber at Durham University, 2011.

1. There are finitely many critical radii r when the homotopy type of $\mathrm{OC}(C; r)$ changes.

2. Small robots: if a radius $r <$ the half-length of a shortest edge, then $\mathrm{OC}(G; r) \sim \mathrm{OC}(G; 0)$.

Homotopy types of $\mathrm{OC}(G; r)$

J. Dover, M. Ozaydin, arXiv1301.5693:

for any $r > 0$, the number of homotopy classes of $\mathrm{OC}_n(G; r)$ is at most quadratic in n, which is a tight upper bound achievable for a star graph.

Similar more general results for isotopy classes of configuration spaces of a metric graph G with restrictions $d(x_i, x_j) \geq r_{ij}$ for $1 \leq i < j \leq n$.

Components of $\mathrm{OC}(G; r)$ for all r

Our problem: design a fast algorithm to find connected components of $\mathrm{OC}(G; r)$ for $r > 0$.

Idea: we reduce the dimension from 2 to 1.

Replace the 2-dimensional geometric space $\mathrm{OC}(G, r)$ by a combinatorial graph that captures all connectivity information of $\mathrm{OC}(G; r)$.

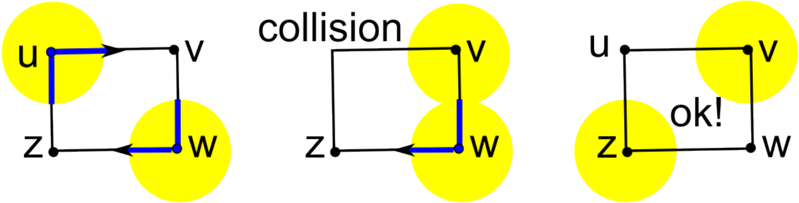
Elementary motion along an edge

Def: let u, v be vertices of an edge $e \subset G$ and a vertex w is away from e, so $d(u, w) \geq 2r$ and $d(v, w) \geq r$. The motion of robot 1 from u to v keeping robot 2 fixed at w is elementary.

If G is a tree, then any collision free motion of 2 robots decomposes into elementary motions.

Elementary motion along a cycle

For graphs with cycles we need 1 more motion.



Def: let u, v, w, z be vertices ordered along a cycle of G, so $d(u, w) \geq 2r > d(u, v)$ and $d(v, z) \geq 2r > d(w, z)$. The motion $u \to v$ for robot 1 and $w \to z$ for robot 2 is elementary.

Any motion $\to$ elementary

Prop: any motion of 2 robots in G can be replaced by a sequence of elementary motions.

Idea: induction on the number of vertices of G that both robots visit during a given motion.

Let both robots start from vertices and robot 1 first reach a new vertex. Then we modify the motion of robot 2 to get a first elementary motion of both robots, and so on.

Configuration skeleton $\mathrm{CS}(G; r)$

Def: the configuration skeleton $\mathrm{CS}(G; r)$ has vertices (u, v) for all vertices $u, v \in G$ with $d(u, v) \geq 2r$. Every edge of $\mathrm{CS}(G; r)$ corresponds to an elementary motion.

Th (Marjan Safi-Samghabadi, VK, 2014):

the inclusion $\mathrm{CS}(G; r) \subset \mathrm{OC}(G; r)$ induces a 1-1 map between connected components.

Algorithm for building $\mathrm{CS}(G; r)$

Input: length matrix of a graph G with k vertices

1. Any edge $\{u, v\}$ and a distant vertex w of G generate an edge $(u, w) \to (v, w)$ of $\mathrm{CS}(G; r)$;

2. Any pair of distant edges $\{u, v\}$ and $\{w, z\}$ in a cycle generate the edge $(u, w) \to (v, z)$.

The overall time so far is $O(k^4)$ for k vertices.

$\mathrm{CS}(G; r)$ *'knows'* if there is a collision-free motion between any configurations $(u, v), (w, z)$.

Summary: spaces of graphs

- the configuration space $\mathrm{OC}(G; r)$ of robots in a metric graph G has a complicated topology depending on G, a radius $r > 0$
- configuration skeleton $\mathrm{CS}(G; r)$ captures all connectivity of the ambient space $\mathrm{OC}(G; r)$
- improve the time for computing $\mathrm{CS}(G; r)$, hence for finding connected components of $\mathrm{OC}(G; r)$, to $O(k^2)$ for G with k vertices.

Problems on configuration spaces

- find the homology and homotopy groups of the configuration space $\mathrm{OC}_n(G; r)$ of robots
- study motions and configurations spaces of round robots in a directed metric graph
- express the maximum radius $R > 0$ when $\mathrm{OC}_n(G; r)$ is connected for all $0 \le r \le R$.
- design an algorithm to find (approximate) a shortest collision-free motion in a metric graph from one configuration to another.