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Gauss paragraphs of classical links and a characterization of virtual link groups

BY VITALIY KURLIN
Department of Mathematical Sciences, Durham University, Durham, DH1 3LE, UK.

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Abstract

A classical link in 3-space can be represented by a Gauss paragraph encoding a link diagram in a combinatorial way. A Gauss paragraph may code not a classical link diagram, but a diagram with virtual crossings. We present a criterion and a linear algorithm detecting whether a Gauss paragraph encodes a classical link. We describe Wirtinger presentations realizable by virtual link groups.

1. *Introduction*

1·1. *Brief summary*

This is a research on the interface between knot theory, group theory and combinatorics. Briefly, a classical knot is a closed loop without self-intersections, considered up to a smooth deformation of the ambient 3-sphere. A classical link is a union of several disjoint closed loops. A plane diagram of a classical link can be combinatorially encoded by a Gauss diagram, a union of circles with arrows. Each arrow connects two points that map to a crossing of the plane diagram. Signs and orientations of arrows allow us to specify the overcrossing information. Plane diagrams of links are defined up to Reidemeister moves, which can be converted into moves on Gauss paragraphs, word codes of Gauss diagrams. A Gauss paragraph may code not a classical link diagram, but a diagram with virtual crossings without specified overcrossing information.

Virtual links generalize classical ones and are defined via plane diagrams with classical and virtual crossings up to natural moves. A virtual link can be considered as a union of linked closed loops in a thickened surface. Virtual crossings invisible on the surface appear in a plane diagram when the surface is projected to a plane. A Wirtinger group has a Wirtinger presentation whose each relation says that two generators are conjugate. The fundamental groups of classical knot complements have Wirtinger presentations with the additional restriction that all generators are conjugate [**3**].

The notion of the fundamental group for classical knot complements extends to a larger class of virtual knots. The resulting Wirtinger groups possess unusual properties, e.g. have non-trivial second homology groups. The groups of virtual knots can be characterized as groups with Wirtinger presentations, where all generators are conjugate and their total number either equals the number of defining relations or exceeds it by 1 [**8**].

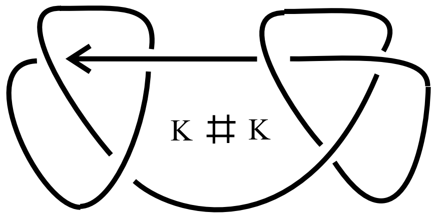


Fig. 1. Non-isotopic connected sums may have isomorphic groups.

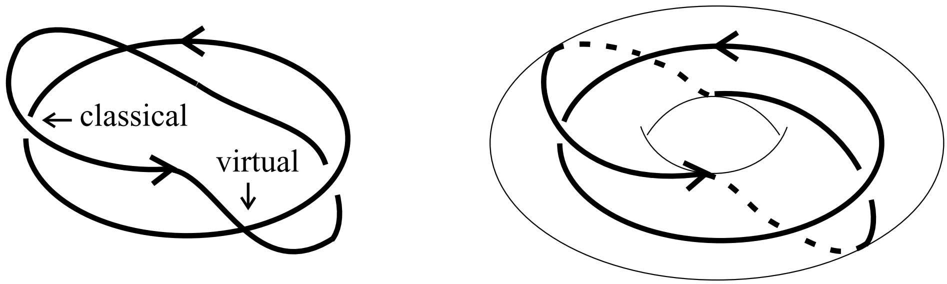


Fig. 2. A virtual link and an embedding into a thickened tours.

1·2. *Classical and virtual links*

Here we introduce basic notions of the classical knot theory and its generalization to virtual links proposed by L. Kauffman in [**7**].

Definition 1·1. A *classical knot* is the image of a smooth embedding $S^1 \to S^3$, i.e. a closed loop without self-intersections, see Figure 1. A *classical link* is a smooth embedding of several disjoint circles.

We will consider oriented links with unordered components. Usually links are studied up to isotopy that is a smooth deformation of S^3, see Definition 2·1. Classical links are represented by plane diagrams defined up to Reidemesiter moves, see Proposition 2·2. The classification problem of knots up to isotopy has an algorithmic solution whose complexity is highly exponential in the number of crossings [**11**]. A polynomial algorithm is not known even for recognizing the *unknot*, a round circle in the plane.

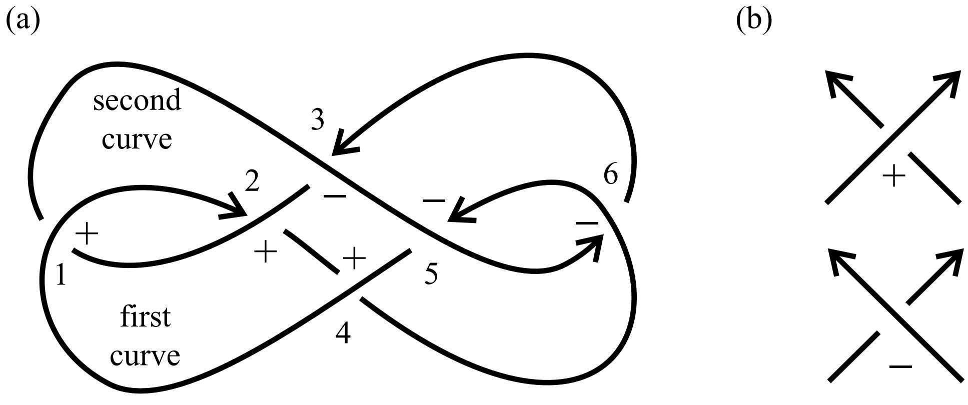
The fundamental groups of knot complements are powerful invariants distinguishing all *prime* knots that are not connected sums of non-trivial knots, see Figure 1. The fundamental group with an additional peripheral structure is a complete knot invariant [**16**]. But this algebraic classification does not provide an effective algorithm for detecting knots.

Definition 1·2. A *virtual* link is a smooth immersion of several oriented circles into the plane, that is a smooth embedding outside finitely many double transversal intersections of 2 types, see Figure 2:

(i) *classical* crossings, where one arc overcrosses the other;
(ii) *virtual* crossings, where the intersecting arcs are not distinguishable.

The *plane* diagram of a virtual link is its image in the plane with the specified overcrossing information at classical crossings only.

Virtual knot theory is motivated by studying knots in thickened surfaces $S_g \times \mathbb{R}$, see [**5**]. Virtual crossings appear under a projection $S_g \to \mathbb{R}^2$. Given a diagram of a virtual knot, the



$\{1\ 2\ 4^+\ 6\ 5^-\ 4,\ 3\ 5\ 6^-\ 3^-\ 2\ 1^+\}$

Fig. 3. (a) A Gauss paragraph of a classical link. (b) Signs of crossings.

genus of S_g is unknown. The equivalence relation for virtual knots is introduced via moves in Definition 4·1.

1·3. *Gauss paragraphs of links*

Firstly we define an abstract Gauss paragraph as a collection of words. After we encode the plane diagram of a classical link by a Gauss paragraph. Our first contribution is Algorithm 1·4 computing the least genus of a thickened surface containing a link encoded by a given Gauss paragraph. We also prove a criterion for the planarity of a Gauss paragraph in Theorem 3·6. Our second contribution is a characterization of Wirtinger presentations realizable as the groups of virtual links, see Theorem 4·8.

Definition 1·3. Given $n > 0$ fix the alphabet $\{i,\ i^+,\ i^- \mid i = 1, \ldots, n\}$. A *Gauss paragraph* is an unordered collection $\{u_1, \ldots, u_k\}$ of words such that the union $u_1 \cup \cdots \cup u_k$ is of even length and contains, for each $i \in \{1, \ldots, n\}$, the letter i and exactly one symbol of the couple i^+, i^-.

Enumerate all crossings in the diagram of a classical oriented link in any order, see Figure 3a. To each curve in the diagram associate a word as follows. Write down the numbers of crossings according to their order in the curve. If we pass through an undercrossing i, we add the superscript ε to the letter i, where ε is the *sign* of the crossing, see Figure 3b. The resulting collection of words is a Gauss paragraph of the plane diagram. The construction is the same for diagrams in oriented surfaces. Any link diagram can be encoded by several Gauss paragraphs since we can renumber the crossings and permute the components.

Any Gauss paragraph gives rise to a link embedded into a thickened surface since any virtual crossing disappears after adding a 1-handle, see Figure 2. In the least genus surface the embedded link is unique up to homeomorphisms of the surface. The virtual knot theory coincides with the theory of links in thickened surfaces up to addition and subtraction of 1-handles outside the link, see [**7**].

ALGORITHM 1·4. *There is an algorithm of linear complexity $O(n)$ to compute the least genus of an oriented surface containing a link diagram encoded by a given Gauss paragraph consisting of* 2*n letters.*

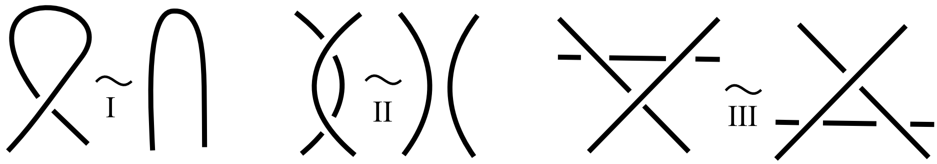


Fig. 4. Classical Reidemeister moves.

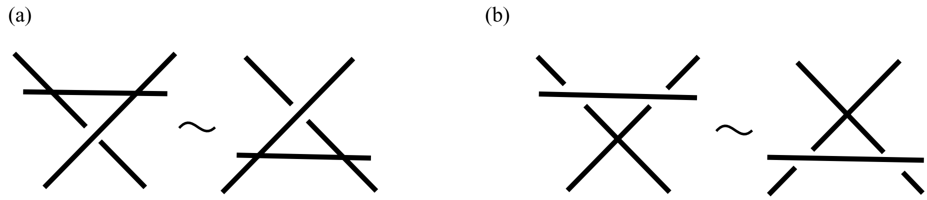


Fig. 5. (a) A mixed move. (b) A forbidden move.

2. *Plane diagrams and Gauss diagrams*

2·1. *An isotopy of knots and Reidemeister moves*

Definition 2·1. Links $K, L \subset S^3$ are called *isotopic* if there is a continuous family of homeomorphisms $F_t\colon S^3 \to S^3$, $t \in [0, 1]$, an *isotopy*, such that $F_0 = \mathrm{id}$ and $F_1(K) = L$.

Plane diagrams of links are supposed to be *regular*, i.e. they have finitely many double transversal intersections, *crossings*. An *isotopy* of diagrams is a smooth family of regular diagrams. In generic isotopies of diagrams only the following codimension 1 singularities can occur: an ordinary cusp ⋎, a simple tangency)(and a transversal triple intersection ✳. The Reidemeister theorem below is an application of singularity theory stating that any isotopy of links can be made transversal to the three codimension 1 discriminants in the space of all smooth links.

PROPOSITION 2·2 (Reidemeister theorem). *Two regular diagrams represent isotopic links if and only if they can be connected by isotopies of diagrams and finitely many* Reidemeister *moves in Figure* 4.

In the above Reidemeister theorem we should consider all symmetric images of the moves in Figure 4 for all possible orientations.

2·2. *Virtual links via plane diagrams*

Firstly we define the equivalence relation for virtual links as in [**7**].

Definition 2·3. Two virtual links are *equivalent* if their plane diagrams can be connected by a finite sequence of the following moves:

(i) three *classical* Reidemeister moves in Figure 4;
(ii) three *virtual* Reidemeister moves, where all crossings are virtual;
(iii) the *mixed* move in Figure 5a with classical and virtual crossings.

The move in Figure 5b and its symmetric images are *forbidden*, because any virtual link can be transformed to the *unlink*, the disjoint union of circles, through the moves of Definition 2·3 and Figure 5b, see [**4**]. The moves of Definition 2·3 generate formally a new equivalence relation on classical link diagrams without virtual crossings, but the resulting theory coincides with the theory of classical links, see [**4**].

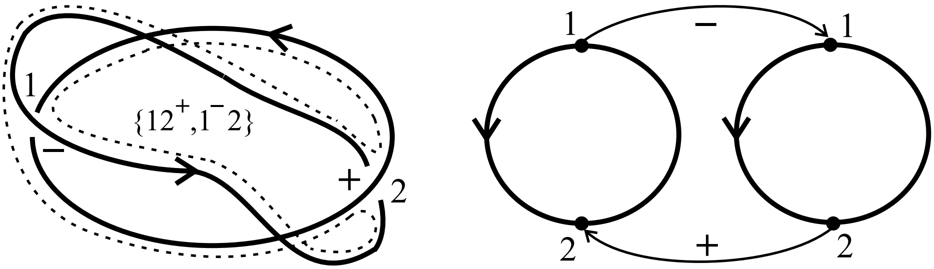


Fig. 6. A virtual 2-component link and its Gauss diagram on 2 circles.

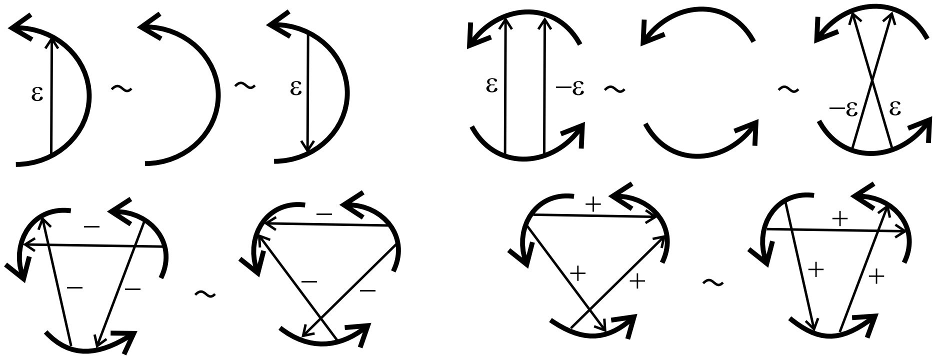


Fig. 7. Moves on Gauss diagrams corresponding to Reidemeister moves.

2·3. *Virtual links via Gauss diagrams*

Now we shall look at virtual links from a purely combinatorial point of view. Virtual links can be represented by Gauss diagrams.

Definition 2·4. A *Gauss* diagram consists of (see Fig. 6):
(i) a union of several oriented circles;
(ii) arrows connecting points on these circles;
(iii) a sign $+$ or $-$ associated to each arrow.
Two Gauss diagrams are considered up to orientation preserving diffeormorphism respecting the arrows and their signs.

Any plane diagram D of a k-component virtual link K gives rise to the following Gauss diagram on k circles. The components of K in the diagram D are immersions of k circles. Two points on these circles are connected by an arrow if and only if they map to a classical crossing in D. The arrow is oriented from the upper branch to the lower one in D. Each arrow is equipped with the sign of the crossing, see Figures 3b and 6.

The moves in Figure 4 generate the transformations of Gauss diagrams in Figure 7. The round arcs may belong to different circles. The Reidemeister move III in Figure 4 gives rise to eight transformations with specified orientations. But the six transformations in Figure 7 are sufficient to realize all Reidemeister moves in Figure 4 [**12**].

The transformations in Figure 7 can be realized by the moves of Definition 2·3 on plane diagrams. The virtual Reidemeister moves and the mixed move in Figure 5a were designed so that any arc involving virtual crossings only can be replaced in a given diagram by an arbitrary arc having the same endpoints and intersecting other arcs in virtual crossings. So

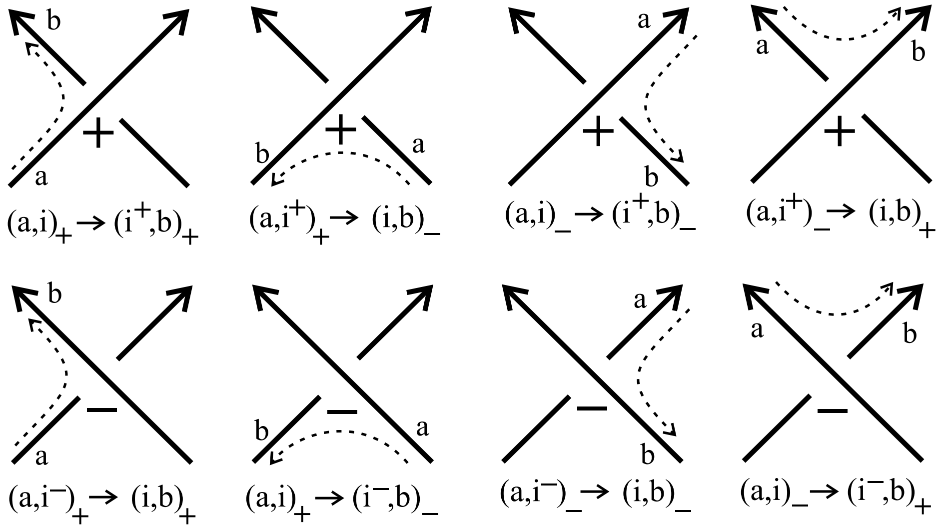


Fig. 8. A geometric interpretation of local rules for selecting cycles.

plane diagrams of virtual links up to the moves of Definition 2·3 are equivalent to Gauss diagrams up to the transformations in Figure 7.

Any Gauss diagram on k circles can be converted into a Gauss paragraph as follows. Number the arrows of the Gauss diagram by $1, \dots, n$ in any order. Label the tail and head of the ith arrow with a sign ε by i and i^{ε}, respectively. By reading the labels counterclockwisely we obtain words $u_1, \dots, u_k$ considered cyclically, see Figure 6.

3. *The planarity of Gauss paragraphs*

3·1. *A linear algorithm for the planarity of Gauss paragraphs*

Any Gauss paragraph can be realized by a diagram with classical crossings on a suitable oriented surface. So any Gauss paragraph of k words encodes an embedding of k circles into a thickened surface. We shall compute the least genus of such a surface in linear time with respect to the length of the Gauss paragraph. If the genus is 0, Algorithm 1·4 determines whether a Gauss paragraph encodes a classical link.

Definition 3·1. Let $\{u_1, \dots, u_k\}$ be a Gauss paragraph consisting of $2n$ letters. We shall construct its *Carter surface* $M\{u_1, \dots, u_k\}$ as a combinatorial cell complex [**2**]. Take n vertices labelled by $1, \dots, n$. We connect vertices i, j by an edge with a mark (a, b) or (b, a), if one of the cyclic words $u_1, \dots, u_k$ contains the ordered pair ab or ba of successive letters, respectively, for some $a \in \{i, i^+, i^-\}$, $b \in \{j, j^+, j^-\}$.

For instance, the Gauss paragraph $\{12^+, 1^-2\}$ in Figure 6 generates the graph with 2 vertices connected by 4 edges (12^+), (2^+1), (1^-2), (21^-).

Let us select unoriented cycles in the resulting graph. Travelling along an edge (a, b), encode the direction of our path by $(a, b)_+$ if the letter a preceeds b in the Gauss paragraph, otherwise by $(a, b)_-$. After passing an edge, we choose the next one by the following local rules (figure 8): $(a, i)_+ \to (i^{\delta}, b)_{\delta}$, $(a, i)_- \to (i^{\delta}, b)_{-\delta}$, $(a, i^+)_{\varepsilon} \to (i, b)_{-\varepsilon}$, $(a, i^-)_{\varepsilon} \to (i, b)_{\varepsilon}$ for a unique possible choice of $\delta = \pm$ and both values of $\varepsilon = \pm$.

A geometric interpretation of the above pure combinatorial rules is in Figure 8. If the given Gauss paragraph encodes a diagram on an oriented surface, then the orders of the letters in the cyclic words correspond to the orientations of the components. Geometrically, the rules say that we always turn left. After passing through each edge once in both directions, we have selected all cycles defining faces in the Carter surface.

The Gauss paragraph $\{12^+, 1^-2\}$ in Figure 6 leads to 2 faces bounded by the cycles $(12^+)_+(21^-)_-(12^+)_-(21^-)_+$ and $(2^+1)_+(1^-2)_-(2^+1)_-(1^-2)_+$. The first cycle is shown by the dashed closed curve in Figure 6. So the Carter surface $M\{12^+, 1^-2\}$ has Euler characteristic 0 and is a torus.

LEMMA 3·2. *Given a Gauss paragraph, the Carter surface has the least genus among all oriented surfaces containing a link diagram encoded by the given Gauss paragraph. A Gauss paragraph encodes a classical link if and only if the Carter surface is a sphere.*

Proof. The Carter surface of the given Gauss paragraph contains a desired link diagram due to the geometric interpretation of Definition 3·1 in Figure 8. Suppose that there is another surface S_g containing a link diagram encoded by the Gauss paragraph. The underlying graph of the diagram with classical crossings divides S_g into several connected pieces. The genus g is minimal, if all the pieces are disks as in Definition 3·1.

Proof of Algorithm 1·4. Assume that a Gauss paragraph $\{u_1, \ldots, u_k\}$ encodes a connected Gauss diagram. Denote by $|u_1|, \ldots, |u_k|$ the lengths of the words. By the rules of Definition 3·1 the genus g the Carter surface $M\{u_1, \ldots, u_k\}$ can be computed in linear time with respect to $2n = |u_1| + \cdots + |u_k|$ via $\chi(M\{u_1, \ldots, u_k\}) = n - \sum_{i=1}^{k} |u_i| + \#(\text{faces}) = 2 - 2g$. By Lemma 3·2 the resulting genus g is minimal. If $g = 0$, the Gauss paragraph represents a classical link. If the Gauss paragraph splits into disjoint subparagraphs, run the algorithm for each of them.

3·2. *A criterion for the planarity of Gauss paragraphs*

To formulate the criterion we first define abstract Gauss codes and then associate a Gauss code to each Gauss paragraph.

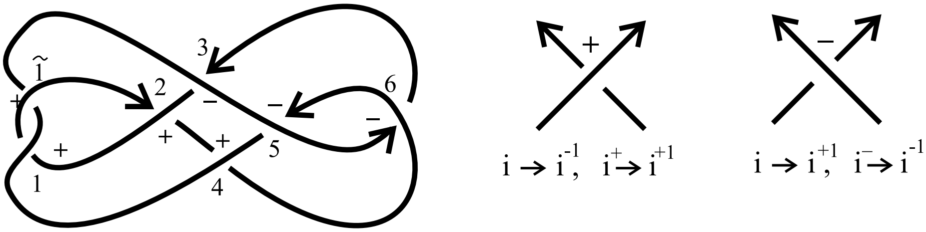
Definition 3·3. A *Gauss code W* is a permutation of $1^{+1}, 1^{-1}, \ldots, n^{+1}, n^{-1}$, considered as a cyclic word. Let S_i be the subword of W between i^{+1} and i^{-1}, not including these symbols. For $i = 1, \ldots, n$, let $\alpha_i(W)$ be the sum of the superscripts of the symbols of S_i. Denote by S_i^{-1} the set of the symbols of S_i, where all superscripts are reversed. Set $\bar{S}_i = S_i \cup \{i^{+1}, i^{-1}\}$. For $i, j \in \{1, \ldots, n\}$, let $\beta_{ij}(W)$ be the sum of the superscripts of the symbols of $\bar{S}_i \cap S_j^{-1}$.

A generic immersion of an oriented circle into the plane can be represented by a Gauss code as follows. Attach indices $1, \ldots, n$ to the crossings in any order, go along the curve and write down the corresponding indices. We add the superscript $+1$ to an index i if the crossing branch at the ith crossing goes from left to right, otherwise -1, see Figure 9.

PROPOSITION 3·4 ([**1**]). *A Gauss code W encodes a planar oriented closed curve if and only if* $\alpha_i(W) = \beta_{ij}(W) = 0$ *for all* $i, j \in \{1, \ldots, n\}$.

For simplicity we assume that a Gauss paragraph can not be split into a disjoint union of non-empty subparagraphs. Construction 3·5 and Theorem 3·6 extends trivially to the case of splittable paragraphs.

CONSTRUCTION 3·5. *To a Gauss paragraph* $\{u_1, \ldots, u_k\}$ *consisting of* $2n$ *letters we associate a Gauss code W of length* $2n + 2k - 2$. *Take two words, say* u_1, u_2, *such that* u_1 *contains a letter i and* u_2 *has* i^+ *or* i^-.



$$\{1\ \tilde{1}^{+}3\ 5\ 6^{-}3^{-}2\ 2^{+}4^{+}6\ 5^{-}4\} \to 1^{-1}\tilde{1}^{+1}3^{+1}5^{+1}6^{-1}3^{-1}2^{-1}1^{+1}\tilde{1}^{-1}2^{+1}4^{+1}6^{+1}5^{-1}4^{-1}$$

Fig. 9. Converting a Gauss paragraph into a Gauss code.

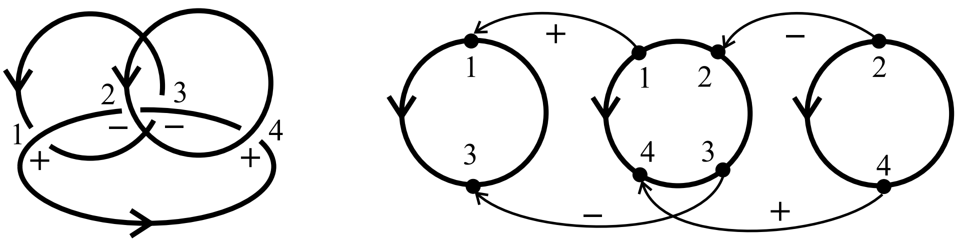


Fig. 10. A non-planar virtual link and its non-planar Gauss diagram.

Rewrite u_2 cyclically in such a way that i^+ or i^- is the last letter of u_2, say i^+. Double these leters by the rule: $i \mapsto i\tilde{i}^+$ and $i^+ \mapsto i^+\tilde{i}$. If the sign of the ith crossing is negative, use the rules $i \mapsto i\tilde{i}^-$ and $i^- \mapsto i^-\tilde{i}$. Now insert the word u_2 ending by $i^+\tilde{i}$ right after $i\tilde{i}^+$ into the word u_1. We get a new word of length $|u_1| + |u_2| + 2$. The Gauss paragraph of 2 *words in Figure* 3 *becomes a single word in Figure* 9 *at the left.*

Continue uniting words until we get a single word of length $2n + 2k - 2$. Finally replace letters as follows: $i^+ \to i^{+1}$, $i^- \to i^{-1}$, $i \to i^{-\varepsilon}$, where ε is the superscript of the other letter with the same index i, see Figure 9.

A Gauss code W associated to a given Gauss paragraph is not uniquely defined, but the following criterion works always.

THEOREM 3·6. *A Gauss paragraph $\{u_1, \ldots, u_k\}$ consisting of $2n$ letters encodes the plane diagram of a classical k-component link if and only if the invariants α_i and β_{ij}, $i, j = 1, \ldots, n$, vanish for any Gauss code W associated to the given Gauss paragraph in Construction* 3·5.

Proof. We prove that the given Gauss paragraph is planar if and only if the Gauss code obtained via Construction 3·5 is planar.

Let D_U and D_W be the plane diagrams encoded by the Gauss paragraph and Gauss code, respectively. If D_U has only classical crossings then D_W is obtained from D_U by doubling $k - 1$ crossings, where different components intersect. Conversely, if D_W has only classical crossings then D_U is obtained from D_W by compressing $k - 1$ pairs of crossings into $k - 1$ single crossings. It remains to apply Proposition 3·4.

More complicated criteria for the planarity of Gauss paragraphs without overcrossing information were obtained in [**13**, **14**].

Figure 10 contains a virtual analogue of the Brunnian link. Any pair of the 3 components is planar, i.e. the corresponding Gauss diagram on 2 circles encodes a plane diagram with classical crossings only. Theorem 3·6 implies immediately that the whole virtual link on 3

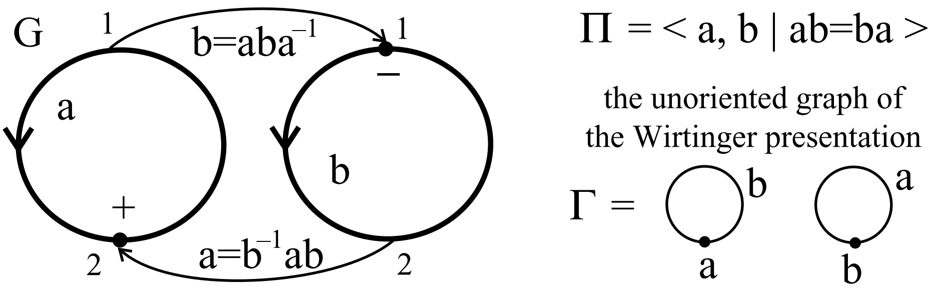


Fig. 11. A Gauss diagram G, the associated presentation Π and its graph Γ.

components is not planar since after doubling crossings 1 and 4, the resulting Gauss code $1^{-1}4^{+1}\tilde{4}^{-1}2^{+1}4^{-1}\tilde{4}^{+1}3^{+1}2^{-1}\tilde{1}^{-1}1^{+1}3^{-1}\tilde{1}^{+1}$ has $\alpha_2 \neq 0$, see Definition 3·3.

4. *The realizability of Wirtinger groups*

4·1. *Virtual link groups*

Firstly we introduce abstract Wirtinger presentations. Then to each virtual link we associate the group with a Wirtinger presentation.

Definition 4·1. A *Wirtinger* group is a group with a *Wirtinger* presentation $\Pi = \langle m_1, \dots, m_n \mid r_1, \dots, r_s \rangle$, where $n > s-1$ and each relator r_q has the form $m_i = w_q^{-1} m_j w_q$ for some $i, j \in \{1, \dots, n\}$ and words w_q in the generators $m_1, \dots, m_n$, $q = 1, \dots, s$. The *weight* of a group is the minimal number k of elements whose conjugates generate the group.

The fundamental group $\pi(L) = \pi_1(S^3 - L)$ of the complement to a classical k-component link always has a Wirtinger presentation [**3**]. The generators of $\pi(L)$ split into k conjugate classes, i.e. the link group $\pi(L)$ is a Wirtinger group of weight k. So the *abelianization* $\pi(L)/[\pi(L), \pi(L)]$ is isomorphic to $\mathbb{Z}^k$ for any k-component link L. We give a general construction for the group of a virtual link from a Gauss diagram.

Definition 4·2. Given a Gauss diagram G with k circles and n arrows, a group Π with n generators and n relations will be constructed. After cutting the circles at each arrowhead and forgetting all arrowtails, G splits into n arcs. To each arc associate a generator of Π, see Figure 11.

Then every arrow gives rise to a defining relation in Π. Let ε be the sign of an arrow in G, the tail of the arrow lie on the arc denoted by a. If the head of the arrow is between the arcs b and c in the counterclockwise direction then the associated relation is $c = a^{-\varepsilon} b a^{\varepsilon}$, see Figures 6, 11.

By Definitions 4·1 and 4·2 the presentation associated to a Gauss diagram is a Wirtinger one. The *group* of a virtual link is the group Π of a Gauss diagram representing the virtual link, see Figure 6. Since generators of Π corresponding to adjacent arcs are conjugate, Lemma 4·3 follows.

LEMMA 4·3. *Let* Π *be the group of a Gauss diagram C on k circles. Then all generators corresponding to the arcs of same circle in C are conjugate to each other. Hence* Π *is normally generated by k elements, the abelianization* $\Pi/[\Pi, \Pi]$ *is isomorphic to* $\mathbb{Z}^k$.

4·2. *The graph of a Wirtinger group*

All generators of a Wirtinger group split into classes of conjugate ones. More explicitly the structure of a Wirtinger group can be described by a finite unoriented graph Γ associated to its presentation Π.

Definition 4·4. A Wirtinger presentation $\Pi = \langle m_1, \dots, m_n \mid r_1, \dots, r_s \rangle$, where $r_q = m_i^{-1} w_q^{-1} m_j w_q$, defines the *graph* Γ as follows. The vertices of Γ are in a 1-1 correspondence with the generators $m_1, \dots, m_n$. Vertices m_i, m_j are connected by an edge marked by w_q if and only if Π has a relator $r_q = m_i^{-1} w_q^{-1} m_j w_q$ for a word w_q in $m_1, \dots, m_n$.

The graph Γ is unoriented and can be disconnected, see the example in Figure 11. The *Euler* characteristic of a connected graph is the number of vertices minus the number of edges, so it is always not bigger than 1.

PROPOSITION 4·5. *Let Γ be the graph of a Wirtinger presentation Π. The connected components of Γ are in a* 1-1 *correspondence with the classes of conjugate generators of Π. If Π is a virtual link group then each component of Γ has Euler characteristic* 0 *or* 1.

Proof. For each relator $r_q = m_i^{-1} w_q^{-1} m_j w_q$ of Π, mark the corresponding edge of Γ by w_q. Any two vertices connected by a path of edges marked by $w_1, \dots, w_p$ correspond to generators conjugate by the product $w_1 w_2 \dots w_p$. Two vertices in Γ are in the same connected component if and only if the corresponding generators of Π are conjugate.

By Definition 4·2 any Gauss diagram provides a Wirtinger presentation, where all generators split into classes of conjugate ones such that within each class the number of generators is equal to the number of relators. Some relators may follow from the remaining ones, as in the case of a classical knot. After removing superfluous relators each connected component of Γ has Euler characteristic either 0 or 1, i.e. either it is a tree or it has exactly one cycle.

4·3. *From a Wirtinger group to a virtual link*

Here we prove that the conditions of Proposition 4·5 are sufficient for the realizability of Gauss paragraphs. In Theorem 4·8 we construct a virtual link starting with a suitable Wirtinger presentation. The method is similar to [**8**, section 3]. The first reduction of Lemma 4·6 converts a given presentation into a cyclic form, where each relator says that two successive generators are conjugate. The second reduction of Lemma 4·7 simplifies all conjugating elements and transforms them to generators.

LEMMA 4·6. *Let Π be a Wirtinger presentation such that each connected component of the associated graph Γ has Euler characteristic* 0 *or* 1. *Then Π can be converted into a presentation $\langle m_1, \dots, m_n \mid r_1, \dots, r_n \rangle$, where each relator has the form $r_i = m_i^{-1} w_i^{-1} m_{i+1} w_i$, $i = 1, \dots, n$, $m_{n+1} = m_1$, for some words $w_1, \dots, w_n$ in $m_1, \dots, m_n$.*

Proof. Let Γ' be a connected component of Γ. If Γ' has Euler characteristic 1 then add a superfluous relation to the presentation to get a cycle (possibly, a loop) in Γ'. We will convert Γ' into a simple cycle, where all vertices have degree 2. After renumbering generators the resulting presentation will have a required cyclic form.

Suppose that Γ' has 2 edges e_{ij}, e_{jk} connecting the vertex i with j, k in such a way that e_{ij} is included into a cycle of Γ', but e_{jk} is not, see Figure 12. Let the corresponding relations of Π be $m_i = w_q^{-1} m_j w_q$ and $m_j = w_p^{-1} m_k w_p$. Then the first relation and edge e_{ij} can be replaced by the relation $m_i = (w_p w_q)^{-1} m_k (w_p w_q)$ and a new edge $e_{ik} \subset \Gamma'$. The procedure works until Γ' is a simple cycle.

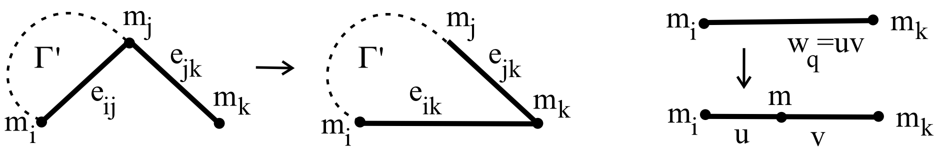


Fig. 12. Simplifying the associated graph Γ of a Wirtinger presentation Π.

The second reduction converts a given presentation into a simple form, where each relator says that two successive generators are conjugate by another generator.

LEMMA 4·7. *Let* Π *be a Wirtinger presenation such that each connected component of the associated graph has Euler characteristic* 0 *or* 1. *Then* Π *can be transformed to a presentation* $\langle m_1, \ldots, m_n \mid r_1, \ldots, r_n \rangle$, *where each relator has the form* $r_i = m_i^{-1} m_q^{-\varepsilon} m_{i+1} m_q^{\varepsilon}$, *where* $\varepsilon = \pm 1$ *and* q *depends on* $i = 1, \ldots, n$.

Proof. We introduce more generators and relators to reduce the words w_i from the given presentation to generators. Assume that the conjugating word w_i can be decomposed as $w_i = uv$ for non-empty words u, v. Then we add a new generator m and replace the given relation $m_i = w_i^{-1} m_j w_i$ by two new ones: $m_i = v^{-1} m v$ and $m = u^{-1} m_j u$.

THEOREM 4·8. *A group can be realized as a virtual link group if and only if it has a Wirtinger presentation such that each connected component of the associated graph has Euler characteristic* 0 *or* 1.

Proof. The necessity was proved in Proposition 4·5. Due to Lemmas 4·6–4·7 it suffices to construct a virtual link whose group has a presentation $\Pi = \langle m_1, \ldots, m_n \mid r_1, \ldots, r_n \rangle$, where each relator has the form $r_i = m_i^{-1} m_q^{-1} m_{i+1} m_q$ for some $q \in \{1, \ldots, n\}$.

For each class of conjugate generators, take a circle and split it into the same number of arcs. Mark the arcs counterclockwisely by the successive generators. For each relator $r_i = m_i^{-1} m_q^{-\varepsilon} m_{i+1} m_q^{\varepsilon}$, draw the arrow from a point on the arc marked by m_q to the common point of m_i, m_{i+1}. The sign of the arrow is ε. By Definition 4·2 the group of this Gauss diagram, i.e. of the required virtual link, has the given presentation Π.

The Gauss paragraph $\{12^+, 1^-2\}$ in Figure 6 is not planar since its Carter surface is a torus. The group of the associated Gauss diagram is isomorphic to $\mathbb{Z}^2$ and is realizable as the group of the classical Hopf link, the graph Γ in Figure 11 satisfies Theorem 4·8.

The criterion of Theorem 4·8 is similar to the characterization of 2-dimensional link groups via their Wirtinger presentations [**6**]. It is known that any abelian group can be realized as the second homology group of the fundamental group of the complement to a knotted surface in 4-sphere [**10**]. The same realizability question for virtual link groups is completely open and striking. There is only one example of a virtual knot group whose second homology group is finite, namely $\mathbb{Z}_2$ [**9**].

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REFERENCES

[**1**] G. CAIRNS and D. M. ELTON. The planarity problem for signed Gauss words. *J. Knot Theory Ramifications* **2** (1993), 359–367.

[**2**] J. S. CARTER. Classifying immersed curves. *Proc. Amer. Math. Soc.* **111** (1991), 281–287.

[3] R. H. CROWELL and R. H. FOX. *Introduction to knot theory*. Graduate Texts in Mathematics, vol. 57, (Springer–Verlag, 1975).

[4] M. GOUSSAROV, M. POLYAK and O. VIRO. Finite type invariants of classical and virtual knots. *Topology* **39** (2000), 1045–1068.

[5] F. JAEGER, L. KAUFFMAN and H. SALEUR. The Conway polynomial in S^3 and thickened surfaces. *J. Combin. Theory Ser. B* **61** (1994), 237–259.

[6] S. KAMADA. A characterization of groups of closed orientable surfaces in 4-space. *Topology* **33** (1994), 113–122.

[7] L. KAUFFMAN. Virtual knot theory. *European J. Combinatorics* **20**, no. 7 (1999), 662–690.

[8] S.-G. KIM. Virtual knot groups and their peripheral structure. *J. Knot Theory Ramifications* **9**, no. 6 (2000), 797–812.

[9] V. KURLIN and D. LINES. Peripherally specified homomorphs of link groups. *J. Knot Theory Ramifications* **16**, no. 6 (2007).

[10] R. A. LITHERLAND. The second homology of the group of a knotted surface. *Quart. J. Math. Oxford* (2) **32** (1981), 425–434.

[11] S. V. MATVEEV. *Algorithmic topology and classification of* 3-*manifolds* (Springer–Verlag, 2003).

[12] O.-P. OSTLUND. Invariants of knot diagrams and relations among Reidemeister moves. *J. Knot Theory Ramifications* **10** (2001), 1215–1227.

[13] P. ROSENSTIEHL. Solution algébrique du problème de Gauss sur la permutation des points d'intersection d'une ou plusieurs courbes fermées du plan. *C.R.Acad. Sci. Paris* **238** (8), (1976), A551–A553.

[14] W. SCHELLHORN. Virtual strings for closed curves with multiple components. math.GT/0412023.

[15] V. VASSILIEV. *Topology of the Complements to Discriminants* (FAZIS, Moscow, 1997).

[16] F. WALDHAUSEN. On irreducible 3-manifolds which are sufficiently large. *Ann of Math.* **87**, no. 1 (1987), 56–88.